

HYPERBOLIC SADDLE MEASURES AND LAMINARITY FOR HOLOMORPHIC ENDOMORPHISMS OF $\mathbb{P}^2\mathbb{C}$

SANDRINE DAURAT

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ABSTRACT. We study the laminarity of the Green current of endomorphisms of $\mathbb{P}^2(\mathbb{C})$ near hyperbolic measures of saddle type. When these measures are supported by attracting sets, we prove that the Green current is laminar in the basin of attraction and we obtain new ergodic properties. This generalizes some results of Bedford and Jonsson on regular polynomial mappings in \mathbb{C}^2 .

1. INTRODUCTION

This article concerns the dynamics of a holomorphic endomorphism f of $\mathbb{P}^2(\mathbb{C})$ (hereafter denoted \mathbb{P}^2). Recall that the Julia set J_1 is the complementary of the regular part of the dynamics and $J_1 = \text{supp}(T)$, where T is the Green current of f . The most chaotic part of the dynamics is a subset J_2 of J_1 that corresponds to the support of the equilibrium measure $\mu_{eq} = T \wedge T$. See [DS], and the references therein, for more results about the dynamics on J_2 and the proprieties of μ_{eq} . The natural measure to consider on J_1 is the trace measure $\sigma_T = T \wedge \omega_{FS}$ of the Green current T which is invariant. We address the question of understanding the behavior of σ_T -almost every point when $J_2 = \text{supp}(\mu_{eq}) \neq J_1$.

We are mainly interested in the case where f admits a trapping region U , i.e. an open set such that $f(U) \Subset U$. The decreasing limit $\mathcal{A} = \bigcap_{n \in \mathbb{N}} f^n(U)$ is called an attracting set. Notice that, since $U \neq \mathbb{P}^2$ is a trapping region, $\text{supp}(\mu_{eq}) \cap U = \emptyset$. If U is Kobayashi hyperbolic then \mathcal{A} is a finite union of attracting periodic orbits, and the dynamics in the basin of \mathcal{A} is well understood. So assume that $U \neq \mathbb{P}^2$ and U is not Kobayashi hyperbolic. In this case, U contains a curve ℓ , see Proposition 2.1.

Bedford-Jonsson [BJ] considered the case of endomorphisms of the form

$$f_0 : [x : y : z] \mapsto [P(x, y, z) : Q(x, y, z) : z^d],$$

near the line at infinity $L_\infty = \{z = 0\}$, which is an attracting set. From the dynamics in \mathbb{P}^1 , it is known that there exists a unique measure μ_∞ of maximal entropy on L_∞ , and μ_∞ represents the equidistribution of saddle points in L_∞ . Moreover, the disintegration of the measure μ_∞ on the unstable manifold L_∞ is induced by T , i.e. $\mu_\infty = T \wedge [L_\infty]$.

In [BJ], Bedford-Jonsson prove that the Green current of f_0 is laminar subordinate to the stable manifolds of μ_∞ in the basin of L_∞ . Thereby, they also obtain that μ_∞ represent the equidistribution of σ_T -almost every points in the basin of L_∞ , see Definition 2.4.

In general, attracting sets have a more complicated structure. In particular, they are generically non-algebraic, see [JW,DT] and the references therein. However, we are going to

see that the dynamics in the basin of attraction is similar to the case describe above. For technical reasons, we assume :

(*Tub*) U is a tubular neighbourhood of a curve ℓ

In particular, U is a euclidean retract¹ of ℓ . We also assume that f satisfies one of the following :

(Sd_t) \mathcal{A} is an attracting set of small topological degree,

or

(SJ) There exists a neighbourhood N of \mathcal{A} in which the Jacobian of f is small, i.e. there exists $0 < \alpha < 1$ such that for all $p = [x : y : z] \in N$, $|\text{Jac}_p(f)| < \alpha \max(|x|, |y|, |z|)^{2d-2}$.

The condition (SJ) is typically satisfied by small perturbations of f_0 , see also section 8. We refer Definition 2.2 for the definition of small topological degree attracting sets and to [Da] and [DT] examples. Under these assumptions, we know

Theorem 1.1 ([Di,Da,DT]). *Let f be an endomorphism of \mathbb{P}^2 admitting a trapping region U . Assume that f and U satisfy the conditions (*Tub*), and (Sd_t) or (SJ). Then*

- (C_1) *there exists a unique invariant current $T^u \in \mathcal{C}_{(1,1)}(U)$, where $\mathcal{C}_{(1,1)}(U)$ is the set of positive closed currents of bidegree $(1,1)$ with supports in U ,*
- (C_2) *$\nu = T \wedge T^u$ is mixing, of entropy $\log(d)$,*
- (C_3) *all measure of entropy $\log(d)$ and support on \mathcal{A} is hyperbolic of saddle type.*

One of the main result of this article is to prove that ν has the same properties as the μ_∞ in [BJ].

Theorem 1.2. *Let f be an endomorphism of \mathbb{P}^2 admitting a trapping region U . Assume that f and U satisfy (*Tub*), and (Sd_t) or (SJ).*

Then the following is true:

- (a) *$\nu = T \wedge T^u$ is the unique measure of maximal entropy $\log(d)$ in $\mathcal{B}_{\mathcal{A}} = \bigcup_{n \in \mathbb{N}} f^{-n}(U)$,*
- (b) *if Per_n denote the set of periodic points of period n then*

$$\nu_n := \frac{1}{d^n} \sum_{\kappa \in \text{Per}_n \cap \mathcal{B}_{\mathcal{A}}} \delta_\kappa \rightarrow \nu, \text{ as } n \rightarrow \infty$$

- (c) *the conditionals of ν on unstable manifolds are induced by T .*

We are going to prove each point of Theorem 1.2 separately under weaker assumptions, see section 2.2. In [DT,Di], the curve ℓ is a line but it can also be a conic as in [FW, section 5]. See also Section 8.1, where we extend the results of [Di] to this setting.

With the conclusions (C_1) and (C_2) of Theorem 1.1, it is quite elementary to prove that there exists an open neighborhood W of \mathcal{A} such that $(f^n)_*(\sigma_T|_W) \rightarrow \nu$ for the weak-* topology. Nevertheless, this does not provide, a priori, a description of the dynamics of σ_T -a.e. point $p \in W$. To this end, we are going to establish the laminarity of the Green current in the basin of attraction.

This question has been studied by several auteurs, see for exemple [deT3,deT4] and [Du1]. Dujardin [Du2] constructed examples of skew-products of \mathbb{C}^2 , that can be extended

¹See [Do, Proposition/Definition IV 8.5]

as endomorphisms of \mathbb{P}^2 , for which the Green current is not laminar near an invariant fibre F that is not attracting.

In [FS], the authors established the laminarity of the Green current in the neighborhood of a uniformly hyperbolic saddle set. We obtain the following result in the non-uniformly hyperbolic case. This generalises [FS] and [BJ] to the basin of an attracting set and, more generally, to the basin of a hyperbolic measure of saddle type.

Theorem 1.3. *Let f be an endomorphism of \mathbb{P}^2 of degree d and T be its Green current. Assume that there exists an invariant current T^u ($\frac{1}{d}f_*T^u = T^u$) such that the measure $\nu = T \wedge T^u$ is ergodic, of entropy $\log(d)$, hyperbolic of saddle type, and $\text{supp}(\nu) \cap \text{supp}(\mu_{eq}) = \emptyset$, where $\mu_{eq} = T \wedge T$ is the equilibrium measure.*

Then there exists a non trivial positive current T^s of bidegree $(1, 1)$ which is laminar and subordinate to the stable manifolds $\bigcup_{x \in \text{supp}(\nu)} W^s(x)$ such that $T^s \leq T$ and for ν -a.e. $x \in \mathcal{A}$, $W^s(x) \subset \text{supp}(T^s)$.

Among other things, Theorem 1.3 gives us information on the topological structure of the Julia set $J_1 = \text{supp}(T)$. We deduce the following:

Corollary 1.4. *Let ν be as in Theorem 1.3. The basin $\mathcal{B}_\nu = \left\{ p \mid \frac{1}{n} \sum_{i=0}^n \delta_{f^i(p)} \rightharpoonup \nu \right\}$ of ν is of positive measure for σ_T .*

Let us emphasize that in Theorem 1.3, and its corollary, ν is not necessarily supported on an attracting set. It is not clear, without further assumption, that $T^s = T$ on an open set, or, that the geometric intersection $T^s \dot{\wedge} T^u$ equals $\nu = T \wedge T^u$, see section 8. Adding the conditions (Tub) , and (Sd_t) or (SJ) , to have (C_1) , we may use a push-pull argument and establish the following result.

Theorem 1.5. *Let f be an endomorphism of \mathbb{P}^2 of degree d . If f admits a trapping region U , such that the conditions (Tub) , and (Sd_t) or (SJ) , are satisfied, then the Green current T of f is laminar and subordinate to the stable manifolds $\bigcup_{x \in \text{supp}(\nu)} W^s(x)$ in the basin of attraction $\mathcal{B}_\mathcal{A} = \bigcup_{n \geq 0} f^{-n}(U)$.*

Corollary 1.6. *Let f be as in Theorem 1.5 and ν be the measure in (C_2) . For σ_T almost every point $p \in \mathcal{B}_\mathcal{A}$*

$$\frac{1}{n} \sum_{i=0}^n \delta_{f^i(p)} \rightharpoonup \nu.$$

The difficulty in Theorem 1.3 is to prove that T^s has positive mass. The particular case where the line at infinity L_∞ is an attracting set and $\nu = T \wedge [L_\infty]$ was handled by E. Bedford and M. Jonsson [BJ]. They use L_∞ as a global transversal to bound this mass from below. Here, we use instead ideas of [BLS], to get the holonomy invariance along stable manifolds and the disintegration of ν on local unstable manifolds.

In [deT3, DDG3], the laminarity of the Green current is obtained by controlling the genus of the curves $f^{-n}L$, where L is a line such that $\frac{1}{d^n}f^{n*}[L] \rightarrow T$. It seems that these arguments do not apply in this setting. See Section 8 for more details.

This article is organized as follows. We start by recalling some facts about laminar currents and Pesin theory in our context. We then study the geometric structure of $\nu = T \wedge T^u$, in particular its disintegration on local unstable manifolds.

Section 4 is devoted to the construction of the current T^s from Theorem 1.3. We then prove Theorem 1.5. In section 6, we establish the equidistribution of periodic points in U on ν . The uniqueness of the measure of maximal entropy is proved in section 7. Theorem 1.2 is a consequence of Proposition 3.3, Corollary 5.2, Theorem 6.4 and Theorem 7.1. We end this paper with some remarks and open questions.

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2. PRELIMINARY

2.1. Attracting set. Let f be an endomorphisms of \mathbb{P}^2 .

Proposition 2.1. *Let \mathcal{A} be an attracting set for f then either \mathcal{A} is trivial, i.e. $\mathcal{A} = \mathbb{P}^2$ or \mathcal{A} is a finite union of attracting periodic orbits, or all trapping region of \mathcal{A} contains a curve.*

Proof. Let U be a trapping region of \mathcal{A} . If $U = \mathbb{P}^2$ or U is Kobayashi hyperbolic then \mathcal{A} is trivial. Assume that U is not Kobayashi hyperbolic, then there exists a positive closed current τ of bidegree $(1, 1)$ with support on U . Since U is an open set, by [G, Theorem 0.1], U contains curves that approximate τ . \square

We also recall the definition of small topological degree attracting sets.

Definition 2.2. *An attracting set \mathcal{A} is said to be of **small topological degree** if there exists a trapping region U and $n \in \mathbb{N}$ such that*

$$\forall p \in f^n(U), \text{card}(f^{-n}(p) \cap U) < d^n.$$

2.2. Assumptions. As mentioned in the introduction, we are going to prove each item of Theorem 1.2 and Theorem 1.5 under weaker assumptions. Here is the different hypotheses we will use:

(H_0) U is a trapping region such that $U \neq \mathbb{P}^2$ and U is not Kobayashi hyperbolic.

We will always denote by $\mathcal{A} = \bigcap_{n \in \mathbb{N}} f^n(U)$ the attracting set associated to U .

(ER) U is a euclidean retract² of ℓ ,

(CV) there exists a unique invariant current $T^u \in \mathcal{C}_{(1,1)}(U)$, where $\mathcal{C}_{(1,1)}(U)$ is the set of positive closed currents of bidegree $(1, 1)$ with supports in U .

Or the weaker version:

(CV^*) There exists a current $T^u \in C_{(1,1)}(U)$ such that for all ϕ $(1,1)$ -form with continuous coefficients and support in $\mathcal{B}_{\mathcal{A}}$ we have $\frac{1}{d^n} f_*^n \phi \rightarrow \langle T, \phi \rangle T^u$.

(H_1) $\nu = T \wedge T^u$ is mixing, of entropy $\log(d)$ and hyperbolic of saddle type.

Or the weaker version:

(H_1^*) $\nu = T \wedge T^u$ is ergodique, of entropy $\log(d)$ and hyperbolic of saddle type.

²See [Do, Proposition/Definition IV 8.5]

(H_2) All measures of entropy $\log(d)$ and supports on \mathcal{A} admit a non positive Lyapunov exponent.

All the examples known so far of attracting sets satisfy the conditions (Tub) and (SJ) but we believe that there exists a larger class of attracting sets. For example, the assumption (H_2) is true as soon as the interior of \mathcal{A} is the empty set or

$$\limsup_{n \rightarrow +\infty} \left(\int_U (f^n)^*(\omega^2) \right)^{1/n} < d,$$

see [DT].

2.3. Laminar and woven currents, geometric intersection. We refer to [De, DS] for general results on currents.

Definition 2.3. A current S is **uniformly woven** in an open set U if there exists a constant $C > 0$ such that

$$S|_U = \int [M_a] \, d\lambda(a)$$

where λ is a measure on the set of holomorphic chains M_a of area bounded by C . Such a measure is called a **marking**.

A current S is **uniformly laminar** if for every point $x \in \text{supp}(T)$ there exist two open sets $\mathbb{B}_1 \subset \mathbb{B}_2$ biholomorphic to the bidisk $D \times D$ such that $x \in \mathbb{B}_1 \subset \mathbb{B}_2$ and in good coordinates

$$S|_{\mathbb{B}_1} = \int [M_a \cap \mathbb{B}_1] \, d\lambda(a)$$

where λ is a measure on the disk $\{0\} \times D$ and the M_a are disjoint graphs $M_a = \{(x, f_a(x))\}$ in \mathbb{B}_2 such that $f_a(0) = a$.

Remark. The marking is not unique. In fact, in \mathbb{C}^2 we have

$$\omega = idz \wedge d\bar{z} + idw \wedge d\bar{w} = \frac{1}{2}id(z+w) \wedge d(\overline{z+w}) + \frac{1}{2}id(z-w) \wedge d(\overline{z-w}).$$

Definition 2.4. A **woven (resp. laminar)** current is the non-decreasing limit of currents of the form

$$S_{\mathcal{Q}} = \sum S_{Q_i}$$

where \mathcal{Q} is a partition in disjoint open sets (resp. open sets biholomorphic to the bidisk) Q_i and S_{Q_i} is a uniformly woven (resp. laminar) current in Q_i .

Definition 2.5. Let R, S be two uniformly laminar currents. We say that R, S **intersect correctly** if, for every $x \in \text{supp}(S) \cap \text{supp}(R)$ there exists an open set U biholomorphic to the bidisk such that R, S are equal to $R|_U = \int [M_a] \, d\lambda(a)$ and $S|_U = \int [N_{a'}] \, d\sigma(a')$, all intersection of the graphs $M_a \cap N_{a'}$ are open sets in M_a and in $N_{a'}$ and $M_a \cap \partial N_{a'}$ (resp. $\partial M_a \cap N_{a'}$) has no mass for $[M_a]$ (resp. $[N_{a'}]$).

Proposition 2.6 ([BLS, Lemma 6.11]). If T_1, \dots, T_n are uniformly laminar currents which intersect correctly then $\max(T_1, \dots, T_n)$ is a well defined laminar current.

Definition 2.7. Let D, D' be two disks. We define the geometrical intersection $D \dot{\wedge} D'$ of D and D' has the sum on the Dirac mass of isolated intersection points, counted with multiplicity, of $D \cap D'$.

Let T_1, T_2 be uniformly woven currents in an open set U and m_1, m_2 be marking of T_1, T_2 . The **geometrical intersection** of T_1 and T_2 is defined by

$$T_1 \dot{\wedge} T_2 = \int [D_1] \wedge [D_2] dm_1 \otimes m_2(D_1, D_2).$$

Remark. Notice that this definition depends, a priori, on the chose of the markings m_1 and m_2 . The following proposition says that the two definitions of intersection coincide, when there are both defined. So, in this case, the geometrical intersection does not depends on the chose of the markings, what should be the general case.

Proposition 2.8. *[DDG2, Proposition 2.6] Let T_1 and T_2 be uniformly woven current, if T_1 or T_2 has bounded potentials, or more generally if $T_1 \in L_{loc}^1(T_2)$ or $T_2 \in L_{loc}^1(T_1)$, then $T_1 \dot{\wedge} T_2 = T_1 \wedge T_2$.*

The geometric intersection can be use to characterize uniformly laminar current and if two uniformly laminar currents intersect correctly.

Proposition 2.9. *Let T_1, T_2 be uniformly woven currents. If $T_1 \dot{\wedge} T_1 = 0$ then T_1 is uniformly laminar and if $T_1 \dot{\wedge} T_2 = 0$ then T_1 and T_2 intersect correctly.*

Proof. The disks in the supports of T_1 and T_2 are holomorhic disks so if $T_1 \dot{\wedge} T_1 = 0$ (resp. $T_1 \dot{\wedge} T_2 = 0$) then for almost every pair (D_1, D_2) of disks in the support of T_1 (resp. in $\text{supp}(T_1) \times \text{supp}(T_2)$) the intersection $D_1 \cap D_2$ is an open set. We conclude using the persistence of the intersection of holomorphic disks. See the beginning of the proof of [Du2, Lemma 2.5]. \square

We may also define the intersection between a positive closed current R of bidegree $(1, 1)$ with bounded potentials and a woven current. In fact, if D is a closed disk in an open set Ω , i.e. $\Omega \cap \partial D = \emptyset$, then $R \wedge [D]$ is well defined in Ω .

Definition 2.10. *Let R be a positive closed current of bidegree $(1, 1)$ with bounded potentials and $S = \int [D] dm(D)$ be a uniformly woven current in an open set $\Omega \subset \mathbb{C}^2$. The **geometrical intersection** of R and S is defined by*

$$R \dot{\wedge} S = \int R \wedge [D] dm(D).$$

Proposition 2.11. *Let R be a positive closed current of bidegree $(1, 1)$ with bounded potentials and $S = \int [D] dm(D)$ be a uniformly woven current in an open set $\Omega \subset \mathbb{C}^2$. If R is also a uniformly woven current in Ω then the definitions 2.7 and 2.10 coincide.*

Proof. This is a direct consequence of [DDG2, Lemma 2.7]. \square

2.4. Pesin Theory and Lyapunov chart. The classical presentation of Pesin theory uses the assumption that $\log(\|Df\|)$ is integrable, see [KH]. We recall here a more general version of it without this assumption.

Let m be an invariant probability measure. Denote by $(\hat{\mathbb{P}}^2, \hat{f}, \hat{m})$ the natural extension of (\mathbb{P}^2, f, m) which is an invertible dynamical system with the same ergodic properties than (\mathbb{P}^2, f, m) . We recall that $\hat{\mathbb{P}}^2 = \{\hat{x} = (x_{-n}) \in (\mathbb{P}^2)^{\mathbb{N}} \mid f(x_{-n}) = x_{-n+1}\}$ is the set of histories with the induced topology of $(\mathbb{P}^2)^{\mathbb{N}}$. The projections π_i and the measure \hat{m} are defined by

$$\pi_i(\hat{x}) = x_i \text{ and } \hat{m}(\hat{A}) = \lim m(\pi_n(\hat{A})).$$

For every point \hat{x} the tangent space is define by $T_{\hat{x}}\hat{\mathbb{P}}^2 = T_{x_0}\mathbb{P}^2 \simeq \mathbb{C}^2$ and we set $D\hat{f}(\hat{x}) = Df(x_0)$, see [deT1] or [Ro].

We start with a more general version of Oseledets Theorem. In the classical statement, we assume that $\log^+(||D\hat{f}^{\pm 1}||) \in L^1(\hat{m})$. As f is an endomorphism of \mathbb{P}^2 , $\log^+(||D\hat{f}||) \in L^1(\hat{m})$. It turns out that the hypothesis $\log^+(||D\hat{f}^{-1}||) \in L^1(\hat{m})$ is not needed, see [Ko, Appendix A1].

Theorem 2.12. *Let f be an endomorphism of \mathbb{P}^2 and m be an invariant measure ($f_*m = m$). There exist a set \hat{X} such that $\hat{m}(\hat{\mathbb{P}}^2 \setminus \hat{X}) = 0$, measurable functions (the Lyapunov exponents) $\chi_1 \geq \chi_2$ and an invariant measurable decomposition of the tangent space such that for all $\hat{x} \in \hat{X}$*

$$T_{\hat{x}}\hat{\mathbb{P}}^2 = E_{\chi_1}(\hat{x}) \oplus E_{\chi_2}(\hat{x})$$

and for all $v \in E_{\chi_i}(\hat{x})$, if $v \neq 0$ then

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log ||D\hat{f}^n(\hat{x})v|| = \chi_i(\hat{x}).$$

The only difference without the assumption $\log^+(||D\hat{f}^{-1}||) \in L^1(m)$, is that the Lyapunov exponents may be equal to $-\infty$.

Remark. If m is ergodic then the Lyapunov exponents are constant.

There also exists a slightly different version of the theorem of γ -reduction of Pesin, without the assumption $\log^+ ||(D\hat{f})^{-1}|| \in L^1(\hat{m})$.

Theorem 2.13. *Let f be an endomorphism of \mathbb{P}^2 and m be an invariant measure ($f_*m = m$). Assume that the Lyapunov exponents of μ satisfies $\chi_u > 0 > \chi_s$. Denote by*

$$E^u(\hat{x}) = E_{\chi_u}(\hat{x}), \text{ and } E^s(\hat{x}) = E_{\chi_s}(\hat{x})$$

For all small enough $\gamma, \varepsilon_0 > 0$, there exist a subset $\hat{Y} \subset \hat{\mathbb{P}}^2$ of full measure and γ -moderate functions $C_\gamma : \hat{Y} \rightarrow GL_2(\mathbb{C})$ and $\delta : \hat{Y} \rightarrow \mathbb{R}$ such that for almost every $\hat{x} \in \hat{Y}$,

- (1) $C_\gamma(\hat{x})$ maps $\mathbb{C}^{\dim E^u(\hat{x})} \oplus \mathbb{C}^{\dim E^s(\hat{x})}$ on the decomposition $E^u(\hat{x}) \oplus E^s(\hat{x})$,
- (2) $g_{\hat{x}}(w) = \text{Exp}_{\gamma, \hat{f}(\hat{x})}^{-1} \circ f_{\hat{x}} \circ \text{Exp}_{\gamma, \hat{x}}(w)$ is well defined on $B(0, \delta(\hat{x}))$, where

$$\text{Exp}_{\gamma, \hat{x}} = \exp_{\hat{x}} \circ C_\gamma(\hat{x}),$$

- (3) $g_{\hat{x}}(0) = 0$,

$$(4) \quad Dg_{\hat{x}}(0) = \begin{pmatrix} A_\gamma^u(\hat{x}) & 0 \\ 0 & A_\gamma^s(\hat{x}) \end{pmatrix} \text{ where } A_\gamma^u \text{ and } A_\gamma^s \text{ are functions such that we have}$$

$$e^{\chi_u(\hat{x})-\gamma} \leq |A_\gamma^u(\hat{x})| \leq e^{\chi_u(\hat{x})+\gamma},$$

$$\text{and } |A_\gamma^s(\hat{x})| \leq e^\alpha \text{ for all } \chi_s(\hat{x}) + \gamma < \alpha < 0,$$

- (5) if we denote $g_{\hat{x}}(w) = Dg_{\hat{x}}(0) + h(w)$ then, for all $||w|| \leq \delta(\hat{x})$, we have

$$||Dh_{\pm}(w)|| \leq \varepsilon_0, \text{ so } ||h_{\pm}(w)|| \leq \varepsilon_0 ||w||.$$

Proof. See [N, Theorem 2.3], Theorem 6.1, and Lemma 6.2 in [Dup] with the notations $\gamma = \varepsilon$ et $\delta(\hat{x}) = \varepsilon_0(\psi_\varepsilon(\hat{f}(\hat{x})))^{-1} = r$. \square

Definition 2.14. We call **horizontal** graph (resp. **vertical** graph), in $\text{Exp}_{\gamma, \hat{x}}B(\delta(\hat{x}))$, the image under $\text{Exp}_{\gamma, \hat{x}}$ of a graph above $B_1(0, \delta(\hat{x}))$ (resp. $B_2(0, \delta(\hat{x}))$).

Proposition 2.15. *Assume that for m-almost every \hat{x} , the Lyapunov exponents satisfy $\chi_u(\hat{x}) > 0 > \chi_s(\hat{x})$. Denote by $B_u(0, r), B_s(0, r)$ the open balls of centre 0 and radius r in E^u and E^s , and denote $B(r) = B_u(0, r) \times B_s(0, r)$.*

Then, for all $0 < r$, $g_{\hat{x}} : B(\rho(\hat{x})) \rightarrow B(\rho(\hat{f}(\hat{x})))$, where $\rho(\hat{x}) = \min(r, \delta(\hat{x}))$, is a horizontal like map of degree 1 and is injective on the restriction of every cut-off horizontal graph, i.e. a horizontal graph in $B(\rho(\hat{x})) \cap g_{\hat{x}}^{-1}(B(\rho(\hat{f}(\hat{x}))))$.

Proof. Let $0 < r \leq \delta(\hat{x})$, up to divided $\delta(\hat{x})$ by a constant, we may assume that Theorem 2.13 is true in $B(r)$. Thus $g_{\hat{x}}(0) = 0$, $Dg_{\hat{x}}(0)$ is a diagonal matrix and for all $w \in B(\delta(\hat{x}))$ we have

$$\|Dg_{\hat{x}}(w) - Dg_{\hat{x}}(0)\| \leq \varepsilon_0.$$

As $\chi_u > 0$ and $\chi_s < 0$, up to reduce $\delta(\hat{x})$ and γ , we may assume that

$$e^{\chi_u - \gamma} - \varepsilon_0 > e^\gamma > 1 \text{ and } e^{\chi_s + \gamma} + \varepsilon_0 < e^{-\gamma} < 1.$$

Denote by p_u the projection on E^u and $h(w) = g_{\hat{x}}(w) - Dg_{\hat{x}}(0)$ so for every $w \in B(r)$

$$|p_u(g_{\hat{x}}(w))| \geq \|Dg_{\hat{x}}(p_u(w))\| - \|h(w)\| \geq e^{\chi_u - \gamma}|p_u(w)| - \varepsilon_0 r \geq e^\gamma r$$

and

$$|g_{\hat{x}}(w)| \leq \|Dg_{\hat{x}}(w)\| + \|h(w)\| \leq e^{\chi_s + \gamma} r + \varepsilon_0 r \leq e^{-\gamma} r.$$

If $\rho(\hat{x}) = \min(r, \delta(\hat{x}))$ then

$$e^\gamma \rho(\hat{x}) \geq \rho(\hat{f}(\hat{x})) \geq e^{-\gamma} \rho(\hat{x}),$$

and by the preceding facts, $g_{\hat{x}} : B(\rho(\hat{x})) \rightarrow B(\rho(\hat{f}(\hat{x})))$ is a horizontal map.

Let Γ be a horizontal graph, i.e. $\Gamma = \{(z, \varphi(z))\}$ where $\varphi : B_u(\rho(\hat{x})) \rightarrow B_s(\rho(\hat{x}))$ is a holomorphic function. We set

$$c = 1 - (\varepsilon_0 + e^\gamma)e^{-\chi_u + \gamma} \text{ and } r' = \rho(\hat{x})(\varepsilon_0 + e^\gamma)e^{-\chi_u + \gamma}$$

so, thanks to the Cauchy inequalities, for all $z, z' \in B^u(r')$ we have $|\varphi(z) - \varphi(z')| \leq \frac{1}{c}|z - z'|$. The image by f of the part of the graph above $B_u(\rho(\hat{x})) \setminus B_u(r')$ is outside $B(\rho(\hat{f}(\hat{x})))$, thus we are interested only in the part of the graph above $B_u(r')$.

Let $w, w' \in \Gamma$ and $z, z' \in B_u(r')$ such that $w = (z, \varphi(z))$, $w' = (z', \varphi(z'))$. Since $Dg_{\hat{x}}(0)$ is diagonal, we have :

$$\|h(w) - h(w')\| \leq 2\varepsilon_0 \max(|z - z'|, |\varphi(z) - \varphi(z')|) \leq \frac{2}{c}\varepsilon_0 |z - z'|$$

and

$$\begin{aligned} |p(g_{\hat{x}}(w)) - p(g_{\hat{x}}(w'))| &\geq \|Dg_{\hat{x}}(0)(z - z', 0)\| - \|h(w) - h(w')\| \\ &\geq \left(e^{\chi_u - \gamma} - \frac{2}{c}\varepsilon_0\right) |z - z'| \end{aligned}$$

And if γ and ε_0 are small enough

$$e^{\chi_u - \gamma} - \frac{2}{c}\varepsilon_0 = e^{\chi_u - \gamma} - \frac{2\varepsilon_0}{1 - (\varepsilon_0 + e^\gamma)e^{-\chi_u + \gamma}} = \frac{e^{\chi_u - \gamma} - e^\gamma - 3\varepsilon_0}{1 - (\varepsilon_0 + e^\gamma)e^{-\chi_u + \gamma}} > 0.$$

Thereby, up to reduce $\delta(\hat{x})$,

$$g_{\hat{x}} : B(\delta(\hat{x})) \cap g^{-1}(B(\delta(\hat{f}(\hat{x})))) \rightarrow B(\delta(\hat{f}(\hat{x})))$$

is of degree 1 and injective on every cut-off horizontal graph. \square

Remark. By the usual Pesin γ -reduction theorem, see Propositions 9 and 10 of [deT1], if $\log^+ \|(D\hat{f})^{-1}\| \in L^1(\hat{m})$ then $g_{\hat{x}}$ is injective in all $B(\rho(\hat{x}))$.

Definition 2.16. For $n \geq 1$, the **graph transform map**

$$f_n : \bigcap_{i=0, \dots, n+1} f^{-i} \left(\text{Exp}_{\gamma, \hat{f}^i(\hat{x})} B(\delta(\hat{f}^i(\hat{x}))) \right) \rightarrow \text{Exp}_{\gamma, \hat{f}^{n+1}(\hat{x})} B(\delta(\hat{f}^{n+1}(\hat{x})))$$

is the composition of n cut-off maps $f|_{\text{Exp}_{\gamma, \hat{f}^i(\hat{x})} B(\delta(\hat{f}^i(\hat{x})))}$ where at each step we cut-off the image of $\text{Exp}_{\gamma, \hat{f}^i(\hat{x})} B(\delta(\hat{f}^i(\hat{x})))$ to $\text{Exp}_{\gamma, \hat{f}^{i+1}(\hat{x})} B(\delta(\hat{f}^{i+1}(\hat{x})))$.

Proposition 2.17. For every $n \geq 1$, the preimage under f_n of a vertical graph is a vertical graph and the image by f_n of a horizontal graph is a horizontal graph.

Moreover, on every horizontal graph the inverse maps f_{-n} of f_n is well defined.

Proof. This is a direct consequence of the preceding proposition. \square

Proposition 2.18. Assume that for m -almost every \hat{x} , the Lyapunov exponent satisfy $\chi_u(\hat{x}) > 0 > \chi_s(\hat{x})$. Then for every $\hat{x} \in \hat{Y}$, there exist a unique local stable manifold $W_{loc}^s(\hat{x})$ and a unique local unstable manifold $W_{loc}^u(\hat{x})$ which are respectively vertical and horizontal graph of $\text{Exp}_{\gamma, \hat{x}} B(\delta(\hat{x}))$. Moreover, there exists a constant $c > 0$ such that

$$\text{for all } y \in W_{loc}^s(\hat{x}), d(f^n(y), f^n(x_0)) \leq ce^{\chi_s n}$$

$$\text{for all } y \in W_{loc}^u(\hat{x}), d(f_{-n}(y), x_{-n}) \leq ce^{-\chi_u n}$$

Proof. This follows from Proposition 2.15. \square

We also recall the concept of common Lyapunov chart, from [BLS], which will have an important role (see also [DDG3]).

Definition 2.19. The **Lyapunov chart** $L(\hat{p})$ of \hat{p} is the image by $\text{Exp}_{\gamma, \hat{p}}$ of the affine bidisk of size $r(\hat{p})$ and axes $E^u(\hat{p}), E^s(\hat{p})$.

Notice that $f : L(\hat{p}) \rightarrow L(\hat{f}(\hat{p}))$ is a horizontal-like map of degree 1. We consider the sets:

$$(1) \quad L_n^s(\hat{p}) := \{y \in L(\hat{p}) \mid \forall 1 \leq j \leq n, f^j(y) \in L(\hat{f}^j(\hat{p}))\} \text{ and } L_n^u(\hat{p}) := f^n L_n^s(\hat{f}^{-n}(\hat{p})),$$

and there analogues in $\hat{\mathbb{P}}^2$

$$\hat{L}_n^s(\hat{p}) := \{\hat{y} \in \pi_0^{-1}(L(\hat{p})) \mid \forall 1 \leq j \leq n, f^j(y_0) \in L(\hat{f}^j(\hat{p}))\} \text{ and } \hat{L}_n^u(\hat{p}) := \hat{f}^n \hat{L}_n^s(\hat{f}^{-n}(\hat{p})).$$

The set $L_n^{s/u}(\hat{p})$ converge exponentially fast to the **local stable/unstable submanifold** $W_{loc}^{s/u}(\hat{p})$. The local stable/unstable submanifold $\hat{W}_{loc}^{s/u}(\hat{p})$ of \hat{p} in $\hat{\mathbb{P}}^2$ is the limit of $\hat{L}_n^{s/u}(\hat{p})$. Thus, depending on the context, we will see the local stable/unstable submanifold either as subset of \mathbb{P}^2 or of $\hat{\mathbb{P}}^2$. Notice that $W^s(\hat{p})$ depend only on $p = \pi_0(\hat{p})$ while $W^u(\hat{p})$ depend on the chose of the preimages of p .

Lemma 2.20. For every $\varepsilon > 0$, there exist a compact set $\hat{\mathcal{R}}_\varepsilon \subset \hat{\mathcal{R}}$ of \hat{m} -measure at least $1 - \varepsilon$, and $r > 0$ such that for every $\hat{p} \in \hat{\mathcal{R}}_\varepsilon$ the submanifold $W_{loc}^s(p)$ (resp. $W_{loc}^u(\hat{p})$) is a graph above a disk of radius $r(p) \geq r$ in $E^s(p)$ (resp. of radius $r(\hat{p}) \geq r$ in $E^u(\hat{p})$) and the restriction to $\hat{\mathcal{R}}_\varepsilon$ of all the preceding maps, as the directions of E^u, E^s and the stable and unstable submanifolds, are continuous.

If \hat{p} and \hat{q} are close enough in $\hat{\mathbb{P}}^2$ then the intersection $\hat{W}_{loc}^s(p) \cap W_{loc}^u(\hat{q})$ is reduced to a point usually denote by $[p, \hat{q}]$. A subset is said to have a product structure if it is closed for $[\cdot, \cdot]$. A **Pesin box** \hat{P} is a compact subset of $\hat{\mathbb{P}}^2$ of positive measure, with a product structure such that the size of the Lyapunov chart of every $\hat{x} \in \hat{P}$ is bounded from below by a positive constant.

Lemma 2.21. (*Bedford-Lyubich-Smillie*) *For each $\eta > 0$, there exists a finite family of disjoint Pesin boxes \hat{P}_i such that $\text{diam}(\pi_0(\hat{P}_i)) < \eta$ and $\bigcup \hat{P}_i$ cover $\hat{\mathcal{H}}_\varepsilon$ (so $\hat{m}(\bigcup \hat{P}_i) \geq 1 - \varepsilon$).*

Proof. The proof of [BLS2, Lemma 1] stay true in our situation, i.e. in $\hat{\mathbb{P}}^2$, up to replace $W_{loc}^{s/u}(\hat{p})$ by $\hat{W}_{loc}^{s/u}(\hat{p})$ \square

Proposition/Definition 2.22. *If $\eta > 0$ is small enough then for every Pesin box \hat{P}_i , satisfying $\text{diam}(\pi_0(\hat{P}_i)) < \eta$, there exists a **common Lyapunov chart** L_i such that $\pi_0(\hat{P}_i) \subset L_i \subset \bigcap_{\hat{x} \in \hat{P}_i} L(\hat{x})$, i.e. for every $\hat{p} \in \hat{P}_i$, $W_{loc}^s(p_0)$ is a vertical disk in L_i and $W_{loc}^u(\hat{p})$ is a horizontal disk in L_i .*

Proof. If $\eta > 0$ is small enough, the stable (resp. unstable) directions of points belonging to \hat{P}_i are almost parallel. Up to reduce η , we assume that $\frac{r}{2} < r - \eta$, thus we chose L_i as the image by $\text{Exp}_{\gamma, \hat{p}}$ of an affine bidisk of axes $E^u(\hat{p}), E^s(p)$ and size $\frac{r}{2}$, where $\hat{p} \in \hat{P}_i$. \square

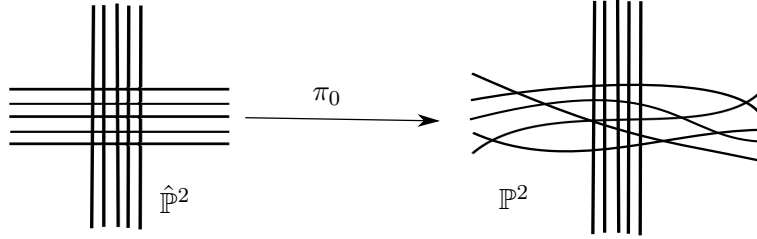


FIGURE 1. The image we have in mind of a Lyapunov chart and the stable/unstable manifolds

Remark. For every $\hat{p} \in \hat{P}_i$, we may replace $L(\hat{p})$ by L_i . This change a little the definitions of $L_n^{s/u}(\hat{p})$ (and of $\hat{L}_n^{s/u}(\hat{p})$) in (1) :

$$(2) \quad \begin{aligned} \forall \hat{p} \in \hat{P}_i, L_n^s(\hat{p}) &:= \{y \in L_i \mid \forall 1 \leq j \leq n, f^j(y) \in L_{k_j}\} \text{ where } p_j \in L_{k_j} \\ \text{and } L_n^u(\hat{p}) &:= f^n L^s(\hat{f}^{-n}(\hat{p})). \end{aligned}$$

For every $\hat{p} \in \hat{P}_i$, denote by $W_{L_i}^{s/u}(\hat{p})$ (or $W_L^{s/u}(\hat{p})$ or $W_{loc}^{s/u}(\hat{p})$) the local stable/unstable manifold

$$(3) \quad W_{L_i}^{s/u}(\hat{p}) = \bigcap_{n \in \mathbb{N}} L_n^{s/u}(\hat{p}),$$

which is a vertical/horizontal disk in L_i .

We recall that $f : L(\hat{p}) \rightarrow L(\hat{f}(\hat{p}))$ is a horizontal-like map of degree 1, so the local stable and unstable manifolds, defined like this, are transverse in each common Lyapunov chart. Notice that if $\log \|d\hat{f}^{-1}\| \notin L^1(\mathfrak{m})$, we cannot assume that $f : L(\hat{p}) \rightarrow L(\hat{f}(\hat{p}))$ is injective.

By the Poincaré recurrence theorem, up to removing a subset of zero measure, we assume that for every $\hat{x} \in \mathcal{R}_\varepsilon$, there exist infinitely many $n > 0$ such that $\hat{f}^n(x) \in \mathcal{R}_\varepsilon$. And, up to remove a subset of ε measure to \mathbb{R}_ε , there exists $C < \infty$ such that, for every \hat{P}_j and every $\hat{x} \in \hat{P}_j$, the following properties are true :

$$(4) \quad \text{dist}(f^n(x_0), f^n(y_0)) < Ce^{-n(\chi^s - \gamma)} \text{ for } n \geq 1 \text{ and } y_0 \in W_{L_j}^s(x_0),$$

$$(5) \quad \text{dist}(x_{-n}, y_{-n}) < Ce^{-n(\chi^u - \gamma)} \text{ for } n > 1 \text{ and } \hat{y} \in W_{L_j}^u(\hat{x}).$$

Let Δ be a disk, it follow from the proof of [FS II, Proposition 5.10] that $T\hat{\lambda}[\Delta] = 0$ iff $(f|_\Delta)^n$ is normal. Thus, for \mathfrak{m} -almost every $\hat{x} \in \mathcal{R}$, $T_{\perp W_{L(\hat{x})}^u(\hat{x})} := T\hat{\lambda}[W_{L(\hat{x})}^u(\hat{x})]$ is a positive measure. Up to remove a subset of ε measure to \mathbb{R}_ε , there exists $m_0 > 0$ such that for every $\hat{x} \in \hat{P}_j \subset \mathbb{R}_\varepsilon$

$$(6) \quad T_{\perp W_{L_j}^u(\hat{x})}(W_{L_j}^u(\hat{x})) \geq m_0.$$

For every $F \subset \hat{P} \subset \mathcal{R}_\varepsilon \subset \{\hat{x} \in \hat{\mathcal{R}} : r(\hat{x}) \geq r\}$, denote by $W_L^{s/u}(F)$ the union of local stable/unstable manifolds of F

$$(7) \quad W_L^{s/u}(F) = \bigcup_{\hat{x} \in F} W_L^{s/u}(\hat{x}).$$

3. CONDITIONAL MEASURES ON UNSTABLE MANIFOLDS

Let \mathfrak{m} be an invariant (under f) probability measure and \mathcal{B} alors $\text{supp}(\mathfrak{m}) \subset \mathcal{A}$. ξ be a measurable partition of \mathbb{P}^2 . The measure \mathfrak{m} may be disintegrated with respect to ξ , i.e. the conditional measures $\mathfrak{m}(\cdot|\xi(x))$ of \mathfrak{m} on the fibers of ξ are well defined, see [BLS] and the references in it.

Denote by $J_{\mathfrak{m},\xi}f$ the Jacobian of f with respect to the conditional measures of \mathfrak{m} on the fibers of ξ , i.e. the Radon-Nikodym derivative of f with respect to $\mathfrak{m}(\cdot|\xi(x))$

$$J_{\mathfrak{m},\xi}f(x) = \frac{df^*\mathfrak{m}(\cdot|\xi(f(x)))}{d\mathfrak{m}(\cdot|\xi(x))} = \frac{1}{\mathfrak{m}(f^{-1}(\xi(f(x)))|\xi(x))}$$

since \mathfrak{m} is invariant under f . Denote by $p(x) = \mathfrak{m}((f^{-1}\xi)(x)|\xi(x))$ and $h_{\mathfrak{m}}(f, \xi)$ the entropy of \mathfrak{m} with respect to ξ , we have:

$$(8) \quad h_{\mathfrak{m}}(f, \xi) = - \int \log(p(x))d\mathfrak{m}(x) = \int \log(J_{\mathfrak{m},\xi}f(x))d\mathfrak{m}(x).$$

The partition ξ is said to be *generating* if the partition $\xi^\infty = \bigvee_{n=0}^\infty f^n\xi$ is the partition generated by singletons. In this case, we have $h_{\mathfrak{m}}(f) = h_{\mathfrak{m}}(f, \xi)$.

Assume that $\text{supp}(\mathfrak{m}) \cap \text{supp}(\mu_{eq}) = \emptyset$ and \mathfrak{m} is of (maximal) entropy $\log(d)$. Then, by Margulis-Ruelle inequality, \mathfrak{m} admit a positive Lyapunov exponent and $\hat{\mathfrak{m}}$ -almost every point $\hat{x} \in \hat{\mathbb{P}}^2$ admit an Pesin unstable manifold $\hat{W}^u(\hat{x})$. Assume also that \mathfrak{m} admit a negative exponent.

Definition 3.1. A measurable partition $\hat{\xi}$ of $\hat{\mathbb{P}}^2$ is said to be subordinate to the unstable manifolds of $(\hat{f}, \hat{\mathfrak{m}})$ if for $\hat{\mathfrak{m}}$ -a.e. $\hat{x} \in \hat{\mathbb{P}}^2$, $\hat{\xi}(x)$ has the following properties:

- (1) $\pi_{0|\hat{\xi}(\hat{x})} : \hat{\xi}(\hat{x}) \rightarrow \pi_0(\hat{\xi}(\hat{x}))$ is bijective,
- (2) $\pi_0(\hat{\xi}(\hat{x})) \subset W^u(\hat{x})$ and $\pi_0(\hat{\xi}(\hat{x}))$ contains an open neighborhood of x_0 in $W^u(\hat{x})$.

Proposition 3.2. *Let m be a f -invariant measure with support outside $\text{supp}(\mu_{eq})$, of (maximal) entropy $\log(d)$. If for m -a.e. x the Lyapunov exponent satisfy $\chi_u(x) > 0 \geq \chi_s(x)$ then there exists a generating partition $\hat{\xi}^u$ of $\hat{\mathbb{P}}^2$ subordinate to Pesin unstable manifolds. Moreover, for all \hat{x} we have*

$$\left(\hat{f}^{-1}\hat{\xi}^u\right)(\hat{x}) = \hat{f}^{-1}\left(\hat{\xi}^u(\hat{f}(\hat{x}))\right) \subset \hat{\xi}^u(\hat{x}).$$

This proposition follows from Proposition 3.2 of [QZ], see also [LS]. In fact, in our case all the properties of the Pesin theory are satisfied except that only the restriction of f to horizontal disks, and not to the entire Lyapunov box, is injective but this is not needed in the proof.

Up to refine $\hat{\xi}^u$, we may assume that for \hat{m} -almost every \hat{x} , $\pi_0(\hat{\xi}(\hat{x}))$ is included in a Lyapunov box. In particular, the map $\pi_0 : \hat{\xi}(\hat{x}) \rightarrow W^u(\hat{x})$ is injective. We have the following proposition analogous to [BLS, Proposition 3.2].

Proposition 3.3. *Let m be a f -invariant measure with support outside $\text{supp}(\mu_{eq})$, of (maximal) entropy $\log(d)$. If for m -a.e. x the Lyapunov exponent satisfy $\chi_u(x) > 0 \geq \chi_s(x)$ then the conditional measures of \hat{m} on $\hat{\xi}^u$ are induced by the Green current T , i.e. for \hat{m} -a.e. \hat{x} we have*

$$\hat{m}(\cdot | \hat{\xi}^u(\hat{x})) = \frac{(\pi_0^{-1})_*(T \wedge [D^u(\hat{x})])}{M\left((\pi_0^{-1})_*(T \wedge [D^u(\hat{x})])\right)}$$

where $D^u(\hat{x}) = \pi_0\left(\hat{\xi}^u(\hat{x})\right)$.

Proof. Let \hat{x} be in the set of Pesin regular points \mathcal{R} . Denote by

$$\rho(\hat{x}) = M\left((\pi_0^{-1})_*(T \wedge [D^u(\hat{x})])\right) = M(T \wedge [D^u(\hat{x})])$$

where $D^u(\hat{x}) = \pi_0\left(\hat{\xi}^u(\hat{x})\right)$. Since $D^u(\hat{x})$ is not a Fatou disk, $M(T \wedge [D^u(\hat{x})]) = \rho(\hat{x}) > 0$, see [FS II, Proposition 5.10]. Thereby we may normalise the family $\left((\pi_0^{-1})_*(T \wedge [D^u(\hat{x})])\right)$ by $\rho(\hat{x})$, to obtain a family of probability measures

$$\mu_{\hat{x}} := \frac{(\pi_0^{-1})_*(T \wedge [D^u(\hat{x})])}{\rho(\hat{x})}.$$

Denote by $q(\hat{x}) = \mu_{\hat{x}} \left(\hat{f}^{-1}(\hat{\xi}^u(f(\hat{x}))) \right)$. By the Pesin theory, $f : f^{-1} \left(W_{loc}^u(\hat{f}(\hat{x})) \right) \cap W_{loc}^u(\hat{x}) \rightarrow W_{loc}^u(\hat{f}(\hat{x}))$ is injective, so we have

$$\begin{aligned} q(\hat{x}) &= \frac{(T \wedge [D^u(\hat{x})]) \left(\pi_0 \left(\hat{f}^{-1}(\hat{\xi}^u(\hat{f}(\hat{x}))) \right) \right)}{\rho(\hat{x})} \\ &= \frac{(T \wedge [D^u(\hat{x})]) \left(f^{-1} \left(\pi_0(\hat{\xi}^u(\hat{f}(\hat{x}))) \right) \right)}{\rho(\hat{x})} \\ &= \frac{1}{d} \cdot \frac{(T \wedge [D^u(\hat{f}(\hat{x}))]) \left(\pi_0(\hat{\xi}^u(\hat{f}(\hat{x}))) \right)}{\rho(\hat{x})} && \text{since } \frac{1}{d} f^* T = T \\ &= \frac{1}{d} \cdot \frac{\rho(\hat{f}(\hat{x}))}{\rho(\hat{x})} \end{aligned}$$

and then

$$\log(q(\hat{x})) = \log(\rho(\hat{f}(\hat{x}))) - \log(\rho(\hat{x})) - \log(d).$$

Since $\hat{f}^{-1}(\hat{\xi}^u(\hat{f}(\hat{x}))) \subset \hat{\xi}^u(\hat{x})$ and $f^* T = dT$, we have

$$M(\rho(\hat{f}(\hat{x}))) = M(T \wedge [D^u(\hat{f}(\hat{x}))]) \leq M(T \wedge [f(D^u(\hat{x}))]) = d M(T \wedge [D^u(\hat{x})]),$$

and thus $\rho(\hat{f}(\hat{x})) \leq d \rho(\hat{x})$ et $q(\hat{x}) \leq 1$. The measure \hat{m} is f -invariant and $\log \circ \rho$ is bounded from above, so

$$(9) \quad - \int \log(q(\hat{x})) d\hat{m}(\hat{x}) = \log(d),$$

see [BLS, Lemma 2.7].

On the other hand, denote by $p(\hat{x}) = \hat{m} \left(\hat{f}^{-1}(\hat{\xi}^u(\hat{f}(\hat{x}))) \mid \hat{\xi}^u(\hat{x}) \right)$, since $\hat{\xi}^u$ is a generator (Proposition 3.2), by (8) we have

$$(10) \quad - \int \log(p(\hat{x})) d\hat{m}(\hat{x}) = h_{\hat{m}}(f, \hat{\xi}^u) = h_{\hat{m}}(f) = \log(d)$$

We deduce from (9) and (10) that

$$\int \log \left(\frac{q(\hat{x})}{p(\hat{x})} \right) = 0.$$

For \hat{m} -almost every \hat{x} we have $\hat{f}^{-1}(\hat{\xi}^u(f(\hat{x}))) \subset \hat{\xi}^u(\hat{x})$ and if $\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \subset \hat{\xi}^u(\hat{x})$ then $\hat{\xi}^u(\hat{y}) = \hat{\xi}^u(\hat{x})$. By definition, p and q are constant on each disk of the form $\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \cap \hat{\xi}^u(\hat{x})$. If $\sum_{\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \subset \hat{\xi}^u(\hat{x})}$ denote the sum indexed on the set of disks of the form

$\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y})))$ included in $\hat{\xi}^u(\hat{x})$ then we have

$$\begin{aligned}
\int \frac{q(\hat{x})}{p(\hat{x})} d\hat{m}(\hat{x}) &= \int \left(\int_{\hat{\xi}^u(\hat{x})} \frac{q(\hat{x})}{p(\hat{x})} d\hat{m}(\hat{x} | \hat{\xi}^u(\hat{x})) \right) d\hat{m}(\hat{x}) \\
&= \int \left(\sum_{\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \subset \hat{\xi}^u(\hat{x})} \int_{\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y})))} \frac{q(\hat{y})}{p(\hat{y})} d\hat{m}(\hat{x} | \hat{\xi}^u(\hat{x})) \right) d\hat{m}(\hat{x}) \\
&= \int \left(\sum_{\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \subset \hat{\xi}^u(\hat{x})} \frac{q(\hat{y})}{p(\hat{y})} \hat{m}(\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) | \hat{\xi}^u(\hat{x})) \right) d\hat{m}(\hat{x}) \\
&= \int \left(\sum_{\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \subset \hat{\xi}^u(\hat{x})} \frac{q(\hat{y})}{p(\hat{y})} \hat{m}(\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) | \hat{\xi}^u(\hat{y})) \right) d\hat{m}(\hat{x}) \\
&= \int \left(\sum_{\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \subset \hat{\xi}^u(\hat{x})} \frac{q(\hat{y})}{p(\hat{y})} \times p(\hat{y}) \right) d\hat{m}(\hat{x}) \\
&= \int \left(\sum_{\hat{f}^{-1}(\hat{\xi}^u(f(\hat{y}))) \subset \hat{\xi}^u(\hat{x})} q(\hat{y}) \right) d\hat{m}(\hat{x}) \\
&= \int 1 d\hat{m}(\hat{x}) \\
&= 1
\end{aligned}$$

Since log is concave, we obtain that for \hat{m} -almost every \hat{x} we have $p(\hat{x}) = q(\hat{x})$, i.e. for \hat{m} -almost every \hat{x}

$$\hat{m}(\hat{f}^{-1}(\hat{\xi}^u(\hat{f}(\hat{x}))) | \hat{\xi}^u(\hat{x})) = \mu_{\hat{x}}(\hat{f}^{-1}(\hat{\xi}^u(f(x)))) .$$

Applying this to \hat{f}^n , and since $\bigvee_{n \in \mathbb{N}} \hat{f}^{-n} \hat{\xi}^u$ is the partition in singletons, we get:

$$\hat{m}(\hat{f}^{-n}(\hat{\xi}^u(\hat{f}^n(\hat{x}))) | \hat{\xi}^u(\hat{x})) = \mu_{\hat{x}}(\hat{f}^{-n}(\hat{\xi}^u(\hat{f}^n(\hat{x}))))$$

and

$$\hat{m}(\cdot | \hat{\xi}^u(\hat{x})) = \mu_{\hat{x}} = \frac{(\pi_0^{-1})_*(T \wedge [D^u(\hat{x})])}{M((\pi_0^{-1})_*(T \wedge [D^u(\hat{x})]))},$$

where $D^u(\hat{x}) = \pi_0(\hat{\xi}^u(\hat{x}))$. □

4. PROOF OF THEOREM 1.3

We start with the construction of a laminar current $T_{\hat{P}}^s \leq T$ subordinate to Pesin unstable manifolds while, in Theorem 1.3, $W^s(x) \subset \text{supp}(T^s)$ for ν -a.e. x . We fix a Pesin box \hat{P}_i and a common Lyapunov chart L_i and denote it by \hat{P} and L .

Theorem 4.1. *If there exists a f -invariant measure ν of entropy $\log(d)$ such that $\text{supp}(\nu) \cap \text{supp}(\mu_{eq}) = \emptyset$ and its Lyapunov exponents satisfy $\chi_u > 0 > \chi_s$ then in each Pesin box \hat{P} , there exists a positive current $T_{\hat{P}}^s$ of bidegree $(1, 1)$, uniformly laminar, of positive mass such*

that $T_{\hat{P}}^s \leq T$ and such that $T_{\hat{P}}^s$ is closed in L . Moreover, $T_{\hat{P}}^s$ is subordinate to $W_L^s(\hat{P})$ and $W_L^s(\hat{P}) = \text{supp}(T_{\hat{P}}^s)$.

Remark. In this theorem we do not assume that ν is ergodic.

By Proposition 3.3, the disintegration of $\hat{\nu}$ on $\hat{\xi}^u$ is induced by the Green current. We are going to prove that they are invariant by holonomy in the common Lyapunov chart L in order to estimate the number of “tubes” of the form L_n^s , and, thereby, assure that $T_{\hat{P}}^s$ has positive mass.

4.1. Holonomy invariance in the common Lyapunov chart L_i . Let \mathcal{W}_L^s be the family of stable manifolds (which are pieces of complex manifolds) of points in \hat{P} , i.e. $\mathcal{W}_L^s = \bigcup_{\hat{p} \in \hat{P}} W_L^s(\hat{p})$, let \mathcal{D}^{horiz} be the set of horizontal disks in L transverse to \mathcal{W}_L^s , i.e. a horizontal disk D is in \mathcal{D}^{horiz} iff D intersects each $W^s \in \mathcal{W}_L^s$ in a unique point, and this intersection is transverse. By the Pesin theory, we have $\bigcup_{\hat{p} \in \hat{P}} W_L^u(\hat{p}) \subset \mathcal{D}^{horiz}$. Let D and D' be two disks in \mathcal{D}^{horiz} , denote by

$$X = \bigcup_{W^s \in \mathcal{W}_L^s} D \cap W^s \text{ and } X' = \bigcup_{W^s \in \mathcal{W}_L^s} D' \cap W^s$$

We define the *holonomy map*

$$\text{hol} := \text{hol}(D, D', \mathcal{W}_L^s) : X \rightarrow X'$$

by $\text{hol}(x) = W^s(x) \cap D'$, where $W^s(x) \in \mathcal{W}_L^s$ is the unique stable manifold containing x .

The holonomy map $\text{hol}(D, D', \mathcal{F}^s)$ is well defined since D and D' are disks in \mathcal{D}^{horiz} and L is a common Lyapunov chart.

Proposition 4.2. *There is holonomy invariance in L , i.e. for all disks D, D' transverse to $W_L^s(\hat{P})$ we have $\text{hol}(T\hat{\lambda}[D]|_X) = T\hat{\lambda}[D]|_{\text{hol}(X)}$.*

Proof. We start with a local proof in a neighborhood of a point $\pi_0(\hat{p})$, where $\hat{p} \in \hat{P}$, and then we use a covering argument to obtain the full result.

Recall that $\hat{p} \in \hat{P} \subset \mathcal{R}_\varepsilon$. Let a, a' be the intersection points $\{a\} = W_L^s(\hat{p}) \cap D$ and $a' = \text{hol}(a) \in W_L^s(\hat{p}) \cap D'$, and, let n be such that $\hat{f}^n(\hat{p}) \in \mathcal{R}_\varepsilon$. Then there exist j such that $\hat{f}^n(\hat{p}) \in P_j$. Denote by $D_n, D'_n \subset L_j$ the cut-off images of D, D' by $f_n : L \rightarrow L_j$. The disks D_n, D'_n are transverse to $\mathcal{W}_{L,n}^s = f_n(\mathcal{W}_L^s)$ and the restriction of f_n to $f_n^{-1}(D_n)$ (resp. $f_n^{-1}(D'_n)$) admits an holomorphic inverse f_{-n} with value in D (resp. D') such that $f_{-n}(D_n) = D \cap L_n^s(\hat{p})$ (resp. $f_{-n}(D'_n) = D' \cap L_n^s(\hat{p})$), see Proposition 2.15. We have $f_{-n}(D_n) = D \cap L_n^s(\hat{p})$ and $f_{-n}(D'_n) = D' \cap L_n^s(\hat{p})$. Denote by

$$\text{hol}_n : X_n \rightarrow X'_n$$

the holonomy map between $X_n = D_n \cap \mathcal{W}_{L,n}^s = f_n(X)$ and $X'_n = D'_n \cap \mathcal{W}_{L,n}^s = f_n(X')$.

Through the end of this section, denote by $r = \delta(\hat{p})$ the “size” of the local stable and unstable manifolds, and $\lambda = \chi_s - \gamma$, see Theorem 2.13. We have the analogous of [BLS, Lemme 4.1]:

Lemma 4.3. *With the preceding notations, we have $\text{hol}_n \circ f_n = f_n \circ \text{hol}$ and for $r_0 < r/4$ $\text{hol}_n(X_n \cap B(f_n(a), r_0 - Ce^{-n\lambda})) \subset \text{hol}_n(X_n) \cap B(f_n(a'), r_0) \subset \text{hol}_n(X_n \cap B(f_n(a), r_0 + Ce^{-n\lambda}))$.*

Since T has a continuous potential, we have:

Lemma 4.4. *If $r/8 < r_0 < r/4$ then there exists a sequence (n_k) of integers such that*

$$\lim \frac{T \wedge D_{n_k}(B(f_{n_k}(a), r_0 \pm Ce^{-n_k \lambda}))}{T \wedge D'_{n_k}(B(f_{n_k}(a'), r_0 \mp Ce^{-n_k \lambda}))} = 1.$$

Proof. The proof follows from [BLS, Lemma 4.2] and (6). \square

To finish the proof of Proposition 4.2, we construct a decreasing family of neighbourhood $C_n^\pm(a)$ and $C'_n(a')$ of $a \in D$ and $a' \in D'$, and use the preceding Lemma as Bedford-Lyubich-Smillie did in the proof of [BLS, Lemma 4.4].

Since $\hat{p}, \hat{f}^n(\hat{p}) \in \mathcal{R}_\varepsilon$, the disks D, D' (resp. D_n, D'_n) are graphs above disks of radius r in $E^u(\hat{p})$ (resp. $E^u(\hat{f}^n(\hat{p}))$).

Let h (resp. h') be holomorphic functions such that D (resp. D') is the image by h (resp. h') of a flat disk. Thanks to the Koebe Distorsion theorem, replacing “ $f^{-n} : W_r^u(f^n(a)) \rightarrow W_r^u(a)$ ” by $f_{-n} : D_n \rightarrow D$ (resp. $f_{-n} : D'_n \rightarrow D'$) in the proof of [BLS, Lemma 4.4], we get that $C_n^\pm(a) = f_{-n}(D_n \cap B(f_n(a), r_0 \pm Ce^{-n\lambda}))$ (resp. $C'_n(a') = f_{-n}(D'_n \cap B(f_n(a'), r_0))$) is the image by h (resp. h') of a convex set.

And thanks to Lemma 4.3, we have $\text{hol}(C_n^-(a) \cap X) \subset C'_n(a) \cap X' \subset \text{hol}(C_n^+(a) \cap X)$. Since $f^n(D_n) = D$, we have $T \wedge [D] = T \wedge f_*^n[D_n] = f_*^n(f^{n*} T \wedge [D_n]) = d^n f_*^n(T \wedge [D_n])$ and we deduce from Lemma 4.4 that

$$(11) \quad \lim \frac{T \wedge D(C_n^\pm(a))}{T \wedge D'(C'_n(a))} = 1.$$

Let $E \subset X$ be a compact set and $E' = \text{hol}(E)$, so for every $\delta > 0$ there exists an open set O of X such that

$$T \wedge D(O) \leq T \wedge D(E) + \delta.$$

The set $\mathcal{C}^\pm = \{C_n^\pm(a) \mid a \in X, n \text{ such that } \hat{f}^n(\hat{p}) \in \mathcal{R}_\varepsilon\}$ (resp. $\mathcal{C}' = \{C'_n(a') \mid a' \in X', n \text{ such that } \hat{f}^n(\hat{p}) \in \mathcal{R}_\varepsilon\}$) is a neighbourhood basis of the points of X in D (resp. of X' in D') and the image by h (resp. h') of convex sets. By the Morse cover theorem [Mo], we deduce that there exists a family $\{C'_j : j = 1, 2, \dots\} \subset \mathcal{C}'$ of non-overlapping open subsets of D' such that

$$(12) \quad T \wedge D' \left(E' - \bigcup_j C'_j \right) = 0.$$

For every $j \in \mathbb{N}$, there exist $a'_j = \text{hol}(a_j)$ and n_j such that $C'_j = C'_{n_j}(a'_j)$ and the corresponding C_j^- (i.e. $C_j^- = C_{n_j}(a_j)$) form also a family of non-overlapping open subsets that belong to \mathcal{C}^- and satisfy $C_j \subset \text{hol}^{-1}(C'_j)$. The diameter of the C'_j can be chosen as small as wanted and hol^{-1} is continuous so we assume that $C_j^- \subset O$, thus

$$T \wedge D(O) \geq \sum_j T \wedge D(C_j^-)$$

and, by (11) and (12), we have

$$T \wedge D(O) \geq \frac{1}{1+\delta} \sum_j T \wedge D'(C'_j) = \frac{1}{1+\delta} T \wedge D'(E').$$

For every $\delta > 0$, we have $T\hat{\Delta}D'(E') + \delta \geq \frac{1}{1+\delta}T\hat{\Delta}D'(E')$, thus

$$T\hat{\Delta}D(E) \geq T\hat{\Delta}D'(E').$$

Similarly, by covering E by a family of \mathcal{C}^+ , we show that $T\hat{\Delta}D'(E') \geq T\hat{\Delta}D(E)$; and conclude that $T\hat{\Delta}D'(E') = T\hat{\Delta}D(E)$. This finish the proof Proposition 4.2. \square

4.2. Number of non-overlapping tubes of the form $L_n^s(\hat{p})$ in L . Up to reduce the size of the common Lyapunov chart L , we assume that $\nu(\partial L) = 0$.

Lemma 4.5. *The set $\mathbb{N}_{\hat{P}} = \{n \in \mathbb{N} \mid \hat{\nu}(\hat{f}^{-n}(\hat{P}) \cap \hat{P}) \geq \hat{\nu}(\hat{P})^2(1 - \varepsilon)\}$ is infinite.*

Proof. Assume that $\mathbb{N}_{\hat{P}}$ is finite, then

$$\frac{1}{m} \sum_{n=0}^{m-1} \hat{\nu}(\hat{f}^{-n}(\hat{P}) \cap \hat{P}) \leq \frac{1}{m} \sum_{\substack{n=0 \\ n \in \mathbb{N}_{\hat{P}}}}^{m-1} \hat{\nu}(\hat{f}^{-n}(\hat{P}) \cap \hat{P}) + \hat{\nu}(\hat{P})^2(1 - \varepsilon),$$

so

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{n=0}^{m-1} \hat{\nu}(\hat{f}^{-n}(\hat{P}) \cap \hat{P}) \leq \hat{\nu}(\hat{P})^2(1 - \varepsilon).$$

On the other hand, as in the proof of [deT2, Lemma 5.1.], we may decompose $\hat{\nu}$ in ergodic measures to get that

$$\hat{\nu}(\hat{P})^2 \leq \limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{n=0}^{m-1} \hat{\nu}(\hat{f}^{-n}(\hat{P}) \cap \hat{P})$$

In fact, this is true for ergodic measures and $x \mapsto x^2$ is convex. We rich a contradiction. \square

Notice that, for every $\hat{p} \in \hat{f}^{-n}(\hat{P}) \cap \hat{P}$, Proposition 4.2 is true by replacing $L(\hat{p})$ by $L_n^s(\hat{p})$

In fact, for every $\hat{q} \in \pi_0^{-1}(L_n^s(\hat{p})) \cap \hat{P}$ the local stable manifold $W_L^s(\hat{q})$ is included in $L_n^s(\hat{p})$, thereby we have holonomy invariance.

Recall the following notations: for every $\hat{p} \in \hat{\mathcal{R}}$, f_{-n} denote the cut-off map on a horizontal disk of $L(\hat{p})$ to a horizontal disk of $L_n^u(\hat{f}^{-n}(\hat{p}))$ and for every subset F of a Pesin box $W^{s/u}(F)$ denote $W^{s/u}(F) = \bigcup_{x \in F} W^{s/u}(x)$.

Lemma 4.6. *The connected component of $L \cap f^{-n}(L)$ which contained $\pi_0(\hat{p})$ is $L_n^s(\hat{p})$. In particular, if $\hat{p}, \hat{q} \in \hat{P} \cap f^{-n}(\hat{P})$ and $L_n^s(\hat{p}) \cap L_n^s(\hat{q})$ is non-empty then $L_n^s(\hat{p}) = L_n^s(\hat{q})$.*

Proof. Assume that there exists a point q which is not in $L_n^s(\hat{p})$ but is in the connected component of $L \cap f^{-n}(L)$ which contained $\pi_0(\hat{p})$. The connected component of $L \cap f^{-1}(L)$ is connected by arcs and contains $L_n^s(\hat{p})$. Let ρ be a path from q to $\pi_0(\hat{p})$ in $L \cap f^{-n}(L)$. Let i be the smallest positive integer such that $f^i(q) \notin L(\hat{f}^i(\hat{p}))$. Since f is a horizontal-like map of degree 1 between the Lyapunov charts, the path $f^i \circ \rho$ cross the vertical side of $L(\hat{f}^i(\hat{p}))$ and so the path $f^n \circ \rho$ cross the vertical side of $L(\hat{f}^n(\hat{p})) = L$. This contradicts the fact that ρ is a path in $L \cap f^{-n}(L)$. \square

Lemma 4.7. *For every Pesin box \hat{P} , $n \in \mathbb{N}$ and $\hat{p} \in \hat{f}^{-n}(\hat{P}) \cap \hat{P}$ we have*

$$\hat{\nu} \left(\pi_0^{-1}(L_n^s(\hat{p})) \cap \hat{P} \cap \hat{f}^{-n}(\hat{P}) \right) \leq d^{-n} \hat{\nu}(\hat{P}).$$

Proof. Denote D, D_n the horizontal disks $D = W_L^u(\hat{p})$ and $D_n = W_L^u(\hat{f}^n(\hat{p}))$. Since $\hat{p} \in \hat{f}^{-n}(\hat{P}) \cap \hat{P}$, we have $f^n(D \cap L_n^s(\hat{p})) = D_n$ so

$$f_*^n(T\lambda[D \cap L_n^s(\hat{p})]) = \frac{1}{d^n} T\lambda f_*^n[D \cap L_n^s(\hat{p})] = \frac{1}{d^n} T\lambda[D_n].$$

By Lemma 4.6, for every $\hat{q} \in \hat{P}$ either $W_L^s(\hat{q}) \cap L_n^s(\hat{p}) = W_L^s(\hat{q})$ or $W_L^s(\hat{q}) \cap L_n^s(\hat{p}) = \emptyset$. Thus, by Proposition 4.2, we have $T\lambda[D_n]|_{\mathcal{W}_L^s} = \text{hol}_{D, D_n} * (T\lambda[D]|_{\mathcal{W}_L^s})$, and

$$f_*^n(T\lambda[D \cap L_n^s(\hat{p})])|_{\mathcal{W}_L^s} = d^{-n} \text{hol}_{D, D_n} * (T\lambda[D]|_{\mathcal{W}_L^s}).$$

Since $\pi_0(\hat{P} \cap \hat{f}^{-n}(\hat{P})) = \pi_0(\hat{f}^{-n}(\hat{P} \cap \hat{f}^n(\hat{P}))) \subset f^{-n}(\pi_0(\hat{P} \cap \hat{f}^n(\hat{P})))$, we have

$$\begin{aligned} T\lambda[D]|_{\mathcal{W}_L^s}(\pi_0(\hat{P} \cap \hat{f}^{-n}(\hat{P})) \cap L_n^s(\hat{p})) &\leq T\lambda[D]|_{\mathcal{W}_L^s}(f^{-n}(\pi_0(\hat{P} \cap \hat{f}^n(\hat{P}))) \cap L_n^s(\hat{p})) \\ &\leq f_*^n(T\lambda[D \cap L_n^s(\hat{p})])|_{\mathcal{W}_L^s}(\pi_0(\hat{P} \cap \hat{f}^n(\hat{P}))) \\ &\leq d^{-n} \text{hol}_{D, D_n} * (T\lambda[D]|_{\mathcal{W}_L^s})(\pi_0(\hat{P} \cap \hat{f}^n(\hat{P}))) \\ &\leq d^{-n} T\lambda[D](\pi_0(\hat{P})). \end{aligned}$$

Notice that $\text{hol}_{D, D_n}^{-1}(\pi_0(\hat{P} \cap \hat{f}^n(\hat{P})))$ is included in $D \cap \pi_0(\hat{P})$ but is not necessarily included in $D \cap \pi_0(\hat{P} \cap \hat{f}^{-n}(\hat{P}))$.

The conditionals of $\hat{\nu}$ with respect to $\hat{\xi}^u$ are induced by T , see Proposition 3.3, thus for every $n \in \mathbb{N}_{\hat{P}}$ and every $\hat{p} \in \hat{f}^{-n}(\hat{P}) \cap \hat{P}$, we have

$$\hat{\nu}(\pi_0^{-1}(L_n^s(\hat{p})) \cap \hat{P} \cap \hat{f}^{-n}(\hat{P}) | \hat{\xi}^u(\hat{p})) \leq d^{-n} \hat{\nu}(\hat{P} | \hat{\xi}^u(\hat{p})),$$

and, thereby, $\hat{\nu}(\pi_0^{-1}(L_n^s(\hat{p})) \cap \hat{P} \cap \hat{f}^{-n}(\hat{P})) \leq d^{-n} \hat{\nu}(\hat{P})$. \square

Proposition 4.8. *For every Pesin box \hat{P} and every $n \in \mathbb{N}_{\hat{P}}$, there is at least $d^n \hat{\nu}(\hat{P})(1 - \varepsilon)$ non-overlapping tubes of the form $L_n^s(\hat{p})$ with $\hat{p} \in \hat{f}^n(\hat{P}) \cap \hat{P}$.*

Proof. For every $\hat{p} \in \hat{P} \cap \hat{f}^{-n}(\hat{P})$ we have $\hat{\nu}(\pi_0^{-1}(L_n^s(\hat{p})) \cap \hat{P} \cap \hat{f}^{-n}(\hat{P})) \leq d^{-n} \hat{\nu}(\hat{P})$. Since

$$\hat{P} \cap \hat{f}^{-n}(\hat{P}) = \bigcup_{\hat{p} \in \hat{P}} \pi_0^{-1}(L_n^s(\hat{p})) \cap \hat{P} \cap \hat{f}^{-n}(\hat{P}),$$

and for every $n \in \mathbb{N}_{\hat{P}}$, we have $\hat{\nu}(\hat{P} \cap \hat{f}^{-n}(\hat{P})) \geq \hat{\nu}(\hat{P})^2(1 - \varepsilon)$, we conclude with Lemma 4.6. \square

4.3. Proof of Theorem 4.1 and a corollary. Recall that \hat{P} is a Pesin box, L is a common Lyapunov chart of \hat{P} and for every $\hat{p} \in \hat{P}$, $L_n^s(\hat{p})$ and $L_n^u(\hat{p})$ are defined by (2).

Proof of Theorem 4.1. Let l be a transverse line to $W_L^u(\hat{P})$ (i.e. to $W_L^u(\hat{p})$ for every $\hat{p} \in \hat{P}$) such that the disk $\Delta = l \cap L$ is vertical. We may chose l such that $\frac{1}{d^n} f_*^n[l] \rightarrow T$.

Fix $\varepsilon > 0$ and $n \in \mathbb{N}_{\hat{P}}$. For every $\hat{p} \in \hat{f}^{-n}(\hat{P}) \cap \hat{P}$, we have $L(\hat{p}) = L(\hat{f}^n(\hat{p})) = L$, so Δ can be seen as a vertical disk of $L(\hat{f}^n(\hat{p}))$. Denote by $\Delta_{n, \hat{p}}$ the cut-off preimage of Δ in

$L_n^s(\hat{p})$, i.e. $\Delta_{n,\hat{p}} = f_{n,\hat{p}}^{-1}(\Delta)$ where $f_{n,\hat{p}} : L_n^s(\hat{p}) \rightarrow L$. By abuse of notation, denote by $f_n^*[\Delta]$ the current

$$f_n^*[\Delta] := \sum_{\hat{p} \in \hat{f}^{-n}(\hat{P}) \cap \hat{P}} f_{n,\hat{p}}^*[\Delta].$$

By definition of $f_{n,\hat{p}} : L_n^s(\hat{p}) \rightarrow L$, we have $\frac{1}{d^n} f_n^*[\Delta] \leq \frac{1}{d^n} f^{n*}[l]$. By Proposition 4.8, $\bigcup_{\hat{p} \in \hat{P} \cap \hat{f}^{-n}(\hat{P})} \Delta_{n,\hat{p}}$ contains at least $d^n \hat{\nu}(\hat{P})(1-\varepsilon)$ disjoint disks, so all cluster values of $\frac{1}{d^n} f_n^*[\Delta]$ is a non trivial positive current, uniformly laminar and is smaller than T . By construction, they are closed in L and subordinate to $W_L^s(\hat{P})$. In fact, $\text{supp}(f_{n,\hat{p}}^*[\Delta]) = \Delta_{n,\hat{p}} \subset L_n^s(\hat{p})$ and $L_n^s(\hat{p})$ converges exponentially fast to $W_L^s(\hat{p})$.

Thereby, all cluster values of $\frac{1}{d^n} f_n^*[\Delta]$ intersect correctly (in L) and are bounded from above by T , so the supremum of these cluster values, denoted by $T_{\hat{P}}^s$, is a well defined laminar current subordinate to $W_L^s(\hat{P})$, see [BLS, Lemma 6.12]. For all $\hat{p} \in \hat{P}$ there exist infinitely many $n \in \mathbb{N}$ such that $\hat{f}^n(\hat{p}) \in \hat{P}$, so $W_L^s(\hat{p})$ is included in the support of at least one cluster value of $\frac{1}{d^n} f_n^*[\Delta]$, thus $W_L^s(\hat{p}) \subset \text{supp}(T_{\hat{P}}^s)$. \square

We deduce the following corollary, see [BLS, Lemma 8.2].

Corollary 4.9. *There exists a continuous psh function $u_{\hat{P}}^s$ defined on L such that for all cut-off function χ with support in L we have $\chi T_{\hat{P}}^s = \chi \text{dd}^c u_{\hat{P}}^s$.*

The product $T_{\hat{P}}^s \wedge T^u$ is well defined in L and $M(T_{\hat{P}}^s \wedge T^u) > 0$.

4.4. Proof of Theorem 1.3.

Proof of Theorem 1.3. Fix a Pesin box \hat{P} . The current $T_{\hat{P}}^s$ of Theorem 4.1 is uniformly laminar, so $\text{supp}\left(\frac{1}{d^n} f^{n*}(T_{\hat{P}}^s)\right) = f^{-n}\left(W_L^s(\hat{P})\right)$. We know that $\text{supp}(\mu_{eq}) \cap W_L^s(\hat{P}) = \emptyset$ and $\text{supp}(\mu_{eq})$ is totally invariant thus $\text{supp}\left(\frac{1}{d^n} f^{n*}(T_{\hat{P}}^s)\right)$ does not intersect $\text{supp}(\mu_{eq})$.

Since $\partial f^{-n}(L)$ is smooth, $T_{\hat{P}}^s \leq T$, and $T \wedge T = 0$ outside $\text{supp}(\mu_{eq})$, we know, by Proposition 2.9, that $\frac{1}{d^n} f^{n*}(T_{\hat{P}}^s)$ is still a uniformly laminar current subordinate to $W^s(\nu) := \bigcup_{x \in \text{supp}(\nu)} W^s(x)$, and the currents $T_{\hat{P}}^s, \dots, \frac{1}{d^n} f^{n*}(T_{\hat{P}}^s)$ intersect correctly. So

$$T_n = \max \left\{ T_{\hat{P}}^s, \dots, \frac{1}{d^n} f^{n*}(T_{\hat{P}}^s) \right\}$$

is a well defined uniformly laminar current, see [BLS, Lemma 6.11], and

$$\text{supp}(T_n) = \bigcup_{i \in \{0, \dots, n\}} \text{supp}(f^{i*}(T_{\hat{P}}^s)).$$

For all $n \in \mathbb{N}$, we have $\frac{1}{d^n} f^{n*}(T_{\hat{P}}^s) \leq T$ so (T_n) is a non-decreasing sequence of uniformly laminar currents bounded by T thus $T^s = \sup_n (T_n)$ is well defined. The current T^s is laminar and subordinate to $W^s(\nu)$, and satisfies $T^s \leq T$. For all $\hat{p} \in \hat{P}$ there exist infinitely many $n \in \mathbb{N}$ such that $\hat{f}^n(\hat{p}) \in \hat{P}$ so $W^s(\hat{p}) \subset \text{supp}(T^s)$.

We may do the same for all Pesin box \hat{P} . Thereby, by taking the supremum on the Pesin boxes, we obtain a laminar current $T^s \leq T$ such that for ν -almost every x , $W^s(x) \subset \text{supp}(T^s)$. \square

5. PROOF OF THEOREM 1.5

We are going to prove Theorem 1.5, and its corollary, under weaker assumptions.

Remark. It is not clear that $\text{supp}(\nu)$ being included in an attracting set is enough to ensure that ν admits a negative Lyapunov exponent.

Theorem 5.1. *Let f be an endomorphism of \mathbb{P}^2 of degree d and T be its Green current. If f admits a trapping region U , such that the conditions (H_0) , (CV^*) and (H_1^*) are satisfied, then T is laminar subordinate to the stable manifolds $\bigcup_{x \in \text{supp}(\nu)} W^s(x)$ in the basin of attraction $\mathcal{B}_{\mathcal{A}} = \bigcup_{n \geq 0} f^{-n}(U)$.*

Corollary 5.2. *Under the assumptions (H_0) , (CV^*) and (H_1^*) , for σ_T -almost every $p \in \mathcal{B}_{\mathcal{A}}$, we have*

$$\frac{1}{n} \sum_{i=0}^n \delta_{f^i(p)} \rightharpoonup \nu.$$

We start with the following proposition:

Proposition 5.3. *In the basin of attraction $\mathcal{B}_{\mathcal{A}}$ of \mathcal{A} , we have*

$$\frac{1}{d^n} f^{n*} T_{\hat{P}}^s \rightarrow cT \text{ where } c = M(T_{\hat{P}}^s \wedge T^u) > 0.$$

Proof. Let ψ be a smooth cut-off function with support in L and ϕ a $(1,1)$ -smooth form with support in $\mathcal{B}_{\mathcal{A}}$, then, by hypothesis (CV^*) , $\frac{1}{d^n} f_*^n \phi \rightarrow \langle T, \phi \rangle T^u$. If the potential $u_{\hat{P}}^s$ of $T_{\hat{P}}^s$ is smooth then

$$\begin{aligned} \left\langle \frac{1}{d^n} f^{n*}(\psi T_{\hat{P}}^s), \phi \right\rangle &= \left\langle \psi T_{\hat{P}}^s, \frac{1}{d^n} f_*^n(\phi) \right\rangle \\ &= \int u_{\hat{P}}^s \left(dd^c(\psi) \wedge \frac{1}{d^n} f_*^n(\phi) + d\psi \wedge \frac{1}{d^n} f_*^n(d^c \phi) + \psi \frac{1}{d^n} f_*^n(dd^c \phi) \right). \end{aligned}$$

Otherwise, since $dd^c(\psi)$, ϕ , $d\psi$, $d^c \phi$, ψ and $dd^c \phi$ are smooth forms and the push forward of a smooth form is a current with continuous coefficients, each term of the sum $dd^c(\psi) \wedge \frac{1}{d^n} f_*^n(\phi) + d\psi \wedge \frac{1}{d^n} f_*^n(d^c \phi) + \psi \frac{1}{d^n} f_*^n(dd^c \phi)$ is well defined and has continuous coefficients. We can define $\frac{1}{d^n} f^{n*}(\psi T_{\hat{P}}^s)$ by

$$\left\langle \frac{1}{d^n} f^{n*}(\psi T_{\hat{P}}^s), \phi \right\rangle = \int u_{\hat{P}}^s \left(dd^c(\psi) \wedge \frac{1}{d^n} f_*^n(\phi) + d\psi \wedge \frac{1}{d^n} f_*^n(d^c \phi) + \psi \frac{1}{d^n} f_*^n(dd^c \phi) \right).$$

Moreover, $u_{\hat{P}}^s$ is continuous so $\phi \mapsto \left\langle \frac{1}{d^n} f^{n*}(\psi T_{\hat{P}}^s), \phi \right\rangle$ is also continuous.

We know that $\|\frac{1}{d^n} f_*^n(d^c \phi)\| \rightarrow 0$ and $\|\frac{1}{d^n} f_*^n(dd^c \phi)\| \rightarrow 0$, see [Di, Proposition 4.7]. We conclude that, for all $(1,1)$ -smooth form ϕ with support in $\mathcal{B}_{\mathcal{A}}$, we have $\left\langle \frac{1}{d^n} f^{n*}(\psi T_{\hat{P}}^s), \phi \right\rangle \rightarrow \langle T, \phi \rangle \langle \psi T_{\hat{P}}^s, T^u \rangle$. \square

We now prove Theorem 5.1.

Proof of Theorem 5.1. As we saw in the proof of Theorem 1.3, $T_n = \max\{T_{\hat{P}}^s, \dots, \frac{1}{d^n} f^{n*}(T_{\hat{P}}^s)\}$ is a well defined uniformly laminar current subordinate to $W^s(\mathcal{A})$. Moreover, $\sup_n(T_n)$ is a well defined laminar current subordinate to $W^s(\mathcal{A})$ and satisfy $\sup_n(T_n) \leq T$. By Proposition 5.3, the non-decreasing limit of (T_n) is equal to cT . Thereby, the restriction of the Green current T to the basin of attraction of \mathcal{A} is a laminar current subordinate to $W^s(\mathcal{A})$. \square

We end this section with the proof of Corollary 5.2. Denote \mathcal{B}_ν the basin of attraction of ν , i.e.

$$\mathcal{B}_\nu := \left\{ p \mid \frac{1}{n} \sum_{i=0}^n \delta_{f^i(p)} \rightharpoonup \nu \right\}.$$

The proof is based on the fact that if $p \in W^s(q)$ then

$$(13) \quad \frac{1}{n} \sum_{i=0}^n \delta_{f^i(p)} - \frac{1}{n} \sum_{i=0}^n \delta_{f^i(q)} \rightharpoonup 0.$$

In fact, let φ be a continuous function defined on $\mathcal{B}_\mathcal{A}$ and let $p, q \in \mathcal{B}_\mathcal{A}$ such that $p \in W^s(q)$, then $\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^n (\varphi \circ f^i(p) - \varphi \circ f^i(q)) = 0$.

Proof of Corollary 5.2. Fix $\varepsilon > 0$ and denote by A_ε the set $A_\varepsilon = \pi_0(\hat{\mathcal{H}}_\varepsilon) \cap \mathcal{B}_\nu$. Let $\hat{P} \subset \hat{\mathcal{H}}_\varepsilon$ be a Pesin box, L be a common Lyapunov chart of \hat{P} and $T_{\hat{P}}^s$ be the current constructed in Theorem 4.1.

We may define the restriction $T_{\hat{P}}^u$ of T^u to $W^u(\hat{P})$. See Section 1.3 and 1.6 of [DDG3] for more details. By [DDG3, Theorem 1.6], $T_{\hat{P}}^u$ is a uniformly woven current. Denote by ν^s, ν^u the measure such that

$$T_{\hat{P}}^s = \int [W_\alpha^s] d\nu^s(\alpha), \quad T_{\hat{P}}^u = \int [\Delta_\beta^u] d\nu^u(\beta)$$

$$\text{and } \nu_{\hat{P}} := T_{\hat{P}}^s \wedge T_{\hat{P}}^u = \int [W_\alpha^s] \wedge [\Delta_\beta^u] d\nu^s(\alpha) \otimes \nu^u(\beta).$$

Since ν is ergodic, by Birkhoff theorem, we have $\nu(\mathcal{B}_\nu) = 1$ so $0 = \nu|_{\hat{P}}(A_\varepsilon^c) \geq \nu_{\hat{P}}(A_\varepsilon^c)$, where $\nu|_{\hat{P}}$ is the restriction of ν to \hat{P} . By Fubini Theorem, for ν^s -a.e. α we have

$$\int [W_\alpha^s] \wedge [\Delta_\beta^u](A_\varepsilon^c) d\nu^u(\beta) = 0.$$

We deduce that for ν^s -a.e. α , we have

$$\int [W_\alpha^s] \wedge [\Delta_\beta^u](A_\varepsilon) d\nu^u(\beta) = \int [W_\alpha^s] \wedge [\Delta_\beta^u](\pi_0(\mathbb{R}_\varepsilon)) d\nu^u(\beta) \geq \int [W_\alpha^s] \wedge [\Delta_\beta^u](\pi_0(\hat{P})) d\nu^u(\beta) > 0.$$

In particular, for ν^s -a.e. α , $W_\alpha^s \cap A_\varepsilon \neq \emptyset$. Since $\sigma_{T_{\hat{P}}^s} = \int [W_\alpha^s] \wedge \omega_{FS} d\nu^s(\alpha)$, for $\sigma_{T_{\hat{P}}^s}$ -a.e. $p \in \mathcal{B}_\mathcal{A}$ there exists α such that $p \in W_\alpha^s$ and $W_\alpha^s \cap A_\varepsilon \neq \emptyset$. Thus there exists $q \in \mathcal{A}_\varepsilon$ such that $p \in W_\alpha^s = W_L^s(q)$. By (13), we have $\frac{1}{n} \sum_{i=0}^n \delta_{f^i(p)} \rightharpoonup \nu$.

Denote $T_n := \max_{1 \leq i \leq n} f^{i*} T_{\hat{P}}^s$, since $f^{-1}(\mathcal{B}_\nu) = \mathcal{B}_\nu$ and $\text{supp}(f^{i*} T_{\hat{P}}^s) = f^{-i} \text{supp}(T_{\hat{P}}^s)$, we have $\sigma_{T_n}(\mathcal{B}_\mathcal{A} \setminus \mathcal{B}_\nu) = 0$. We deduce from the proof of Theorem 1.5 that (σ_{T_n}) is a non-decreasing sequence of measures converging to σ_T thus

$$\sigma_T(\mathcal{B}_\mathcal{A} \setminus \mathcal{B}_\nu) = \lim \sigma_{T_n}(\mathcal{B}_\mathcal{A} \setminus \mathcal{B}_\nu) = 0.$$

Hence for σ_T -a.e. $p \in \mathcal{B}_\mathcal{A}$ we have $\frac{1}{n} \sum_{i=0}^n \delta_{f^i(p)} \rightharpoonup \nu$. □

6. EQUIDISTRIBUTION OF SADDLE PERIODIC POINTS

In this section, we follow the ideas of [BLS] which have also been used in [DDG3]. We assume that f admits an attracting set \mathcal{A} which satisfies the condition (ER) and that there exists an invariant current T^u ($\frac{1}{d}f_*T^u = T^u$) such that the measure $\nu = T \wedge T^u$ satisfies (H_1) .

Remark. In this section, we do not need to have any convergence toward the current T^u .

Denote by $\mathcal{B}_{\mathcal{A}}$ the basin of attraction of \mathcal{A} . Fix a Pesin box \hat{P} and a Lyapunov chart L of \hat{P} .

Lemma 6.1 (Shadowing lemma). *For all $\hat{x} \in \hat{P} \cap \hat{f}^{-n}(\hat{P})$ there exists a (unique) periodic point $\kappa(\hat{x}) \in L_n^s(\hat{x}) \cap L_n^u(\hat{f}^n(\hat{x}))$ of period n .*

Proof. See [BLS2, p.284] or [DDG3, Section 9 Step 2]. \square

Denote by Per_n the set of periodic points of period n and $P_{L,n} = \{\kappa(\hat{p}) \mid \hat{p} \in \hat{P}\}$. For all $\kappa \in P_{L,n}$ denote

$$\Omega(\kappa) = \{\hat{x} \in \hat{P} \cap \hat{f}^{-n}(\hat{P}) \mid \kappa(\hat{x}) = \kappa\}.$$

Let us recall that if $\hat{p}, \hat{q} \in \hat{P} \cap \hat{f}^{-n}(\hat{P})$ then $L_n^s(\hat{p}) \cap L_n^s(\hat{q})$ is empty or $L_n^s(\hat{p}) = L_n^s(\hat{q})$, see Lemma 4.6. So

$$\Omega(\kappa) \subset \hat{P} \cap \hat{f}^{-n}(\hat{P}) \cap \pi_0^{-1}(L_n^s(\kappa)),$$

where $L_n^s(\kappa) = L_n^s(\hat{x})$ if $\kappa(\hat{x}) = \kappa$.

Lemma 6.2. *We have $\liminf_{n \rightarrow \infty} \frac{1}{d^n} \#(Per_n \cap L) \geq \hat{\nu}(\hat{P})$.*

Proof. By Lemma 4.7, we know that for every $\kappa \in P_{L,n}$, $\hat{\nu}(\Omega(\kappa)) \leq d^{-n} \hat{\nu}(\hat{P})$. By Lemma 6.1, $\hat{P} \cap \hat{f}^{-n}(\hat{P})$ is the disjoint union $\bigcup_{\kappa \in P_{L,n}} \Omega(\kappa)$, thus

$$\hat{\nu}(\hat{P} \cap \hat{f}^{-n}(\hat{P})) = \sum_{\kappa \in P_{L,n}} \hat{\nu}(\Omega(\kappa)) \leq \frac{1}{d^n} \hat{\nu}(\hat{P}) \#(P_{L,n})$$

and $d^{-n} \#(P_{L,n}) \geq \frac{\hat{\nu}(\hat{P} \cap \hat{f}^{-n}(\hat{P}))}{\hat{\nu}(\hat{P})}$. Since $P_{L,n} \subset Per_n \cap L$, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{d^n} \#(Per_n \cap L) \geq \frac{\hat{\nu}(\hat{P} \cap \hat{f}^{-n}(\hat{P}))}{\hat{\nu}(\hat{P})}.$$

The measure ν is mixing, we conclude that $\liminf_{n \rightarrow \infty} \frac{1}{d^n} \#(Per_n \cap L) \geq \hat{\nu}(\hat{P})$. \square

Lemma 6.3. *Let*

$$\nu_n = \frac{1}{d^n} \sum_{\kappa \in Per_n \cap U} \delta_{\kappa}$$

and $\tilde{\nu}$ be a cluster value of (ν_n) , then $\tilde{\nu} \geq \nu$.

Proof. Since every set can be cover, up to a ν -null set, by Pesin boxes and $\text{supp}(\nu) \subset \mathcal{B}_{\mathcal{A}}$, the result follows from the previous Lemma. \square

Theorem 6.4. *Let f be an endomorphism of \mathbb{P}^2 which admits an attracting set \mathcal{A} which f admits an attracting set \mathcal{A} . Assume, moreover, \mathcal{A} admits a trapping region satisfying the conditions (ER), and that there exists an invariant current T^u ($\frac{1}{d}f_*T^u = T^u$) with support on \mathcal{A} such that the measure $\nu = T \wedge T^u$ satisfies (H_1) . Then*

$$\nu_n = \frac{1}{d^n} \sum_{\kappa \in \text{Per}_n \cap \mathcal{B}_{\mathcal{A}}} \delta_{\kappa} \rightarrow \nu.$$

Proof. The restriction of f to an invariant curve is of topological entropy $\log(d) > 0$ so f cannot have a curve of fixed points. Since $f(U) \subseteq U$, $\mathcal{A} = \bigcap_{n \in \mathbb{N}} f^n(U)$ and $\mathcal{B}_{\mathcal{A}} = f^{-n}(U)$, there is no fixed points in $\mathcal{B}_{\mathcal{A}} \setminus U$. The compact set \overline{U} is an euclidean retract, see [Do, Proposition/Definition IV 8.5], and $f(\overline{U}) \subseteq \overline{U}$ is compact. Hence by Lefschetz-Hopf theorem, see [Do, Proposition VII 6.5], the number of periodic points of period n in \overline{U} is $\sum \text{Trace} \left((f^n)^*_{|H^i(\overline{U}, \mathbb{Q})} \right)$.

Since \overline{U} retracts on ℓ , $H^i(\overline{U}, \mathbb{Q}) = H^i(\ell, \mathbb{Q}) = H^i(\mathbb{P}^1 \mathbb{C}, \mathbb{Q})$ and for a generic line $\Delta \subset U$ we have $f_*^n \Delta \cdot \Delta = d^n$, thus $\#(\text{Per}_n \cap U) = d^n + 1$. Thereby, every cluster value of $\nu_n = \frac{1}{d^n} \sum_{\kappa \in \text{Per}_n \cap U} \delta_{\kappa}$ has mass 1 and we conclude with Lemma 6.3. \square

7. UNIQUENESS OF THE MEASURE OF MAXIMAL ENTROPY

We assume that f is an endomorphism of \mathbb{P}^2 admitting a non trivial attracting set \mathcal{A} and that conditions (CV), (H_1^*) and (H_2) are satisfied

We still denote by $\mathcal{B}_{\mathcal{A}}$ the basin of attraction of \mathcal{A} , T the Green current of f and T^u the attracting current. The aim of this section is to prove the following Theorem:

Theorem 7.1. *Let f be an endomorphism of \mathbb{P}^2 admitting a non trivial attracting set \mathcal{A} which satisfies the condition (CV), (H_1^*) and (H_2) then $\nu = T \wedge T^u$ is the unique measure of maximal entropy $\log(d)$ in $\mathcal{B}_{\mathcal{A}}$.*

To prove this theorem we follow the approach of [BLS].

Proof of Theorem 7.1. Let m be a f -invariant measure with support in $\mathcal{B}_{\mathcal{A}}$ and of (maximal) entropy $\log(d)$ then, by Choquet representation theorem, ν can be written as an integral of ergodic measures which also are of maximal entropy $\log(d)$, since the metrical entropy is concave. So we only have to prove that ν is the only ergodic measure of maximal entropy in $\mathcal{B}_{\mathcal{A}}$. Let m be an ergodic measure of maximal entropy with support in $\mathcal{B}_{\mathcal{A}}$. By (H_2) , m admits a non positive Lyapunov exponent.

The measure \hat{m} is also ergodic and, by Birkoff theorem, for every continuous function φ and \hat{m} -a.e. \hat{x} we have

$$(14) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\hat{f}^i(\hat{x})) = \int \varphi(\hat{x}) d\hat{m}(\hat{x}).$$

Denote $\hat{m}_{\hat{x}} := \hat{m}(\cdot | \hat{\xi}^u)$ then, by dominate convergence and (14), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\hat{\xi}^u(\hat{x})} \varphi(\hat{y}) d \left(\frac{1}{n} \sum_{i=0}^{n-1} \hat{f}_*^n(\hat{m}_{\hat{x}}) \right) (\hat{y}) &= \lim_{n \rightarrow \infty} \int_{\hat{\xi}^u(\hat{x})} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\hat{f}^n(\hat{y})) d\hat{m}_{\hat{x}}(\hat{y}) \\ &= \int_{\hat{\xi}^u(\hat{x})} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\hat{f}^n(\hat{y})) \right) d\hat{m}_{\hat{x}}(\hat{y}) \\ &= \int \varphi(\hat{y}) d\hat{m}(\hat{y}) \end{aligned}$$

hence

$$(15) \quad \frac{1}{n} \sum_{i=0}^{n-1} \hat{f}_*^n(\hat{m}_{\hat{x}}) \rightharpoonup \hat{m}.$$

On the other hand, for every \hat{x} , since

$$M \left(dd^c \left(\frac{1}{d^n} f_*^n [D^u(\hat{x})] \right) \right) = M \left(\frac{1}{d^n} f_*^n dd^c ([D^u(\hat{x})]) \right) = O \left(\frac{1}{d^n} \right),$$

every cluster value of $\left(\frac{1}{d^n} f_*^n [D^u(\hat{x})] \right)$ is a positive closed current of support $M([D^u(\hat{x})])$ with support in \mathcal{A} , see [Di] for more details. By the condition (CV), we have

$$\frac{1}{d^n} f_*^n [D^u(\hat{x})] \rightarrow c T^u$$

with $c = M([D^u(\hat{x})])$. The Green current T has a continuous potential so

$$\frac{1}{\rho(\hat{x})} T \wedge \left(\frac{1}{d^n} f_*^n [D^u(\hat{x})] \right) \rightarrow \frac{c}{\rho(\hat{x})} T \wedge T^u = \frac{c}{\rho(\hat{x})} \nu.$$

So $c = \rho(\hat{x})$, since $\frac{1}{\rho(\hat{x})} T \wedge [D^u(\hat{x})]$ and ν are probability measures. Thus we have

$$(16) \quad f_*^n \left(\frac{T \wedge [D^u(\hat{x})]}{\rho(\hat{x})} \right) \rightarrow \nu.$$

Thanks to Proposition 3.3, we know that $\hat{m}_{\hat{x}} = (\pi_0^{-1})_* \left(\frac{1}{\rho(\hat{x})} T \wedge [D^u(\hat{x})] \right)$ but (16) is not enough to conclude that

$$\hat{f}_*^n(\hat{m}_{\hat{x}}) = f_*^n \left((\pi_0^{-1})_* \frac{T \wedge [D^u(\hat{x})]}{\rho(\hat{x})} \right) \rightarrow \hat{\nu}.$$

However, for all $\hat{X} \subset \hat{A}$ we have

$$\begin{aligned} \hat{f}_*^n(\hat{m}_{\hat{x}})(\hat{X}) &= \frac{1}{\rho(\hat{x})} T \wedge [D^u(\hat{x})] \left(\pi_0(\hat{f}^{-n}(\hat{X})) \right) \\ &= \frac{1}{\rho(\hat{x})} T \wedge [D^u(\hat{x})] \left(\pi_{-n}(\hat{X}) \right). \end{aligned}$$

If $0 \leq i \leq n$ then $\pi_{-n}(\hat{X}) \subset f^{-(n-i)} \pi_{-i}(\hat{X})$ so

$$\begin{aligned} \hat{f}_*^n(\hat{m}_{\hat{x}})(\hat{X}) &\leq \frac{1}{\rho(\hat{x})} T \wedge [D^u(\hat{x})] \left(f^{-(n-i)} \pi_{-i}(\hat{X}) \right) \\ &\leq \frac{1}{\rho(\hat{x})} T \wedge \frac{1}{d^{-(n-i)}} f_*^{-(n-i)} [D^u(\hat{x})] \left(\pi_{-i}(\hat{X}) \right). \end{aligned}$$

We let n go to the infinity, and, by (15) and (16), we obtain that for every $i \in \mathbb{N}$

$$\hat{m}(\hat{X}) \leq \nu(\pi_{-i}(\hat{X})),$$

so $\hat{m} \leq \hat{\nu}$. But \hat{m} and $\hat{\nu}$ are probability measures so $\hat{m} = \hat{\nu}$, and $m = \nu$. This ends the proof of Theorem 7.1. \square

8. FURTHER REMARKS

8.1. Hypothesis for Theorem 1.5.

8.1.1. Previously known settings. The conditions given in the introduction may do reformulate thanks to [DT, Di, Ta]. In fact, in [Di], T.C. Dinh prove that if U contains an image of $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^2 \setminus U$ is star-shaped, then U support a natural positive closed current T^u of bidegree $(1, 1)$, and $\nu = T \wedge T^u$ is mixing of entropy $\log(d)$.

Remark. It is not clear that the fact that $\text{supp}(\nu)$ is include in an attracting set is enough to ensure that ν admits a negative Lyapunov exponent.

If we further assume that f is of small topological degree on U then f satisfies (CV) and (H_1) , see [DT]. This two conditions are also true in the setting of [Ta], i.e. if for all $x \in U$, $\|D_x f\| < 1$ and there exist a point $I \notin U$ and a line $\ell \subset U$ such that for all $x \in \ell$ the set $I(x) \cap U \subset I(x) \setminus I \simeq \mathbb{C}^2$ is strictly convex, where $I(x)$ is the line passing through I and x .

If instead we further assume that the rational hull $r(K)$ of the compact set $K = \mathbb{P}^2 \setminus U$ (see [G, Definition 2.1]) does not intersect \mathcal{A} , then f satisfies (CV*). This follows from [G, Lemma 2.7] and [Di, Theorem 4.6].

8.1.2. New examples. In practice, the only examples known that satisfy (CV*) (and also all the assumptions in section 2.2) are the perturbations of the line at infinity exposed in the introduction. In this section, we present new examples.

We fix the following notations:

$$F_\theta : [x : y : z] \mapsto [x^2 : y^2 : xy + \theta(z^2 - xy)]$$

$$U = \{[x : y : z] \mid |z^2 - xy| \leq \delta \max(|x|, |y|, |z|)^2\}$$

and $X = x^2, Y = y^2, Z = xy + \theta(z^2 - xy)$. Assume that $0 < |\theta| \leq \theta_0$ and that θ_0 and δ are small, then $[0 : 0 : 1] \notin U$ and for all $[x : y : z] \in U$, $\theta_0 |z^2| \leq \max(|x|^2, |y|^2) \leq \max(|X|, |Y|, |Z|)$. Thus for $[x : y : z] \in U$ we have $\theta_0 \max(|x|, |y|, |z|)^2 \leq \max(|X|, |Y|, |Z|)$ and

$$\begin{aligned} (17) \quad |Z^2 - XY| &= |\theta(z^2 - xy)| |xy - Z| \\ &\leq \delta \theta_0 \max(|x|, |y|, |z|) \cdot 2 \max(|X|, |Y|, |Z|) \\ &\leq 2\delta \max(|X|, |Y|, |Z|)^2 \end{aligned}$$

So $F_\theta(U) \subset\subset U$ and $\bigcap_{n \geq 0} F_\theta^n(U) = \{z^2 = xy\}$, i.e. the conic $\{z^2 = xy\}$ is an attracting set for F_θ .

The trapping region U does not satisfy the hypotheses of Dinh [Di] but we are going to prove :

Proposition 8.1. *If f is an endomorphism of \mathbb{P}^2 such that $U = \{[x : y : z] \mid |z^2 - xy| \leq \delta \max(|x|, |y|, |z|)^2\}$ is a trapping region for f then f satisfy (CV*).*

Remark. Even if it seems possible, it is not clear how to exhibit an example such that no trapping region satisfy Dinh's hypotheses. We may consider the family of maps

$$\tilde{F}_\theta[x : y : z] \mapsto [P(x, y, z)^2 : Q(x, y, z)^2 : P(x, y, z)Q(x, y, z) + \theta(z^2 - xy)^d],$$

where P, Q are homogeneous polynomials of $\mathbb{C}[X, Y, Z]$ of degree $d \geq 1$ such that, for all $\theta \neq 0$, \tilde{F}_θ is an endomorphism of \mathbb{P}^2 , and the indeterminacy points of \tilde{F}_0 are not on $\{z^2 = xy\}$.

Lemma 8.2. *The rational hull $r(K)$ of the compact set $K = \mathbb{P}^2 \setminus U$ (see [G, Definition 2.1]) is equal to K .*

Proof. Let $[a_1 : a_2 : a_3] \in U$ and chose i such that $|a_i| = \max(|a_1|, |a_2|, |a_3|)$, so $|a_1 a_2 - a_3^2| < \varepsilon |a_i|^2$. Denote by \tilde{C} the conic $\tilde{C} = \{[x_1 : x_2 : x_3] : a_i^2(x_1 x_2 - x_3^2) = x_i^2(a_1 a_2 - a_3^2)\}$. Let $[x_1 : x_2 : x_3] \in \tilde{C}$ then

$$|x_1 x_2 - x_3^2| = \frac{|a_1 a_2 - a_3^2|}{|a_i^2|} |x_i|^2 < \varepsilon \max(|x_1|, |x_2|, |x_3|).$$

Thus the conic \tilde{C} is included in U and contains $[a_1 : a_2 : a_3]$. \square

Thanks to [G, Lemma 2.7], we only have to see how to adapt Dinh's proofs to get the conclusion of Theorem 1.1 and Theorem 4.6 of [Di]. Let R be a positive closed current with continuous coefficients and support in U . The only time Dinh uses the assumption on the geometry of U is in the section 3 to construct the structural disks. Here is how we can do it in this situation.

Fix a chart W of $\text{Aut}(\mathbb{P}^2)$ containing id and local holomorphic coordinates A , such that $\|A\| < 1$ and $A = 0$ at id . Denote by :

- $W' \Subset W$ a small neighbourhood of id ,
- U' an open set such that $f(U) \Subset U' \Subset U$,
- V a simply connected neighbourhood of the interval $[0, 1]$ in \mathbb{C} ,
- for $\alpha \in \mathbb{C}$ and $\|A\| \leq \min(1, |1 - \alpha|^{-1})$, $\lambda_\alpha(A) := (1 - \alpha)A$,
- π_1, π_2 the projections of $V \times \mathbb{P}^2$ to the first and second coordinates.

We choose V, W', θ small enough such that for all $\alpha \in V$, $A \in W'$, and all $p \in F_\theta^{-1}(U')$ we have $\lambda_\alpha(A) \circ F_{\alpha\theta}(p) \in U$. For all $A \in W'$, denote by:

$$\begin{aligned} \mathcal{F}_A : V \times \mathbb{P}^2 &\rightarrow V \times \mathbb{P}^2 \\ (\alpha, p) &\mapsto (\alpha, \lambda_\alpha(A) \circ F_{\alpha\theta}(p)) \end{aligned}$$

This is a holomorphic endomorphism outside $\{0\} \times \mathbb{P}^2$. Since $\mathcal{F}(\{0\} \times \mathbb{P}^2) = \{0\} \times A(\mathcal{C})$, we can extend trivially the current

$$\mathcal{R}_A = \frac{1}{d} \mathcal{F}_{A*} \left(\pi_2^* \left(\frac{1}{d} F_\theta^* R \right) \right)$$

to $V \times \mathbb{P}^2$ as a positive closed current, see [HP]. We define $R_{\alpha, A}$ to be the structural discs $R_{\alpha, A} = \langle \mathcal{R}_A, \pi_2, \alpha \rangle$, see [Di, Appendix A] for notations. Let ρ be a smooth positive probability measure with compact support in V and denote

$$R_\alpha = \int R_{\alpha, A} d\rho(A).$$

We have $R_1 = \frac{1}{d} F_{\theta*} \left(\frac{1}{d} F_\theta^* R \right) = R$, since $\lambda_1(A) = \text{id}_{\mathbb{P}^2}$, and $R_0 = \int A_*[\mathcal{C}] d\rho(A)$ is independent of R .

The end of the proof is exactly the same than the one of Dinh.

8.2. Around Theorem 1.3. Let ν be as in theorem 1.3, i.e.

- ν is of the form invariant current $\nu = T \wedge T^u$, where T^u is an invariant current ($\frac{1}{d}f_*T^u = T^u$) and T is the Green current of f ,
- ν is of entropy $\log(d)$ and hyperbolic of saddle type,
- $\text{supp}(\nu) \cap \text{supp}(\mu_{eq}) = \emptyset$, where $\mu_{eq} = T \wedge T$ is the equilibrium measure.

Denote by \mathcal{B}_ν the basin of ν . It follows from Theorem 1.3 that $\sigma_T(\mathcal{B}_\nu) > 0$. A natural question is to know if σ_T almost every point is in the basin of a hyperbolic measure of saddle type.

For every Pesin box \hat{P} we constructed (see Theorem 4.1) a laminar current $T_{\hat{P}}^s$ such that $T_{\hat{P}}^s \wedge T^u \leq T \wedge T^u = \nu$ and $M(T_{\hat{P}}^s \wedge T^u) \geq \nu(\hat{P})(1 - \varepsilon) \cdot \nu^u(\hat{P})$ where ν^u is the marking of the restriction $T_{\hat{P}}^u$ of T^u to $W^u(\hat{P})$, i.e. the measure such that $T_{\hat{P}}^u = \int [\Delta_\beta^u] d\nu^u(\beta)$.

For different reasons we believe that Theorem 4.1 can be improve.

Question. Under the same assumptions, can we construct, and all $\varepsilon > 0$, a uniformly laminar current $T_{\hat{P}}^s$ such that $M(T_{\hat{P}}^s \wedge T^u) \geq \nu(\hat{P})(1 - \varepsilon)$. Or can we construct, for every all $\varepsilon > 0$, a laminar current T_ε^s such that $T_\varepsilon^s \leq T$, and

$$M(T_\varepsilon^s \wedge T^u) \geq 1 - \varepsilon?$$

8.3. The control of the genus of the curves $f^{-n}L$. We mentioned in the introduction that a way to prove that the Green current is laminar is to control the genus of the curves $f^{-n}L$, where L is a line such that $\frac{1}{d^n}f^{n*}[L] \rightarrow T$. In several cases, the growth of the genus of the curves $f^{-n}L$ is linked with the number of preimages of a point; the link is given by the Riemann-Hurwitz formula.

More precisely, by [deT3, Theorem 1], if $\text{genus}(f^{-n}(L) \cap U) = O(d^n)$ then T is laminar. H. de Thelin [deT4] proved that this true for post-critically finite maps. In the general case, he obtained:

Theorem 8.3 ([deT4, Theorem 2]). *For a generic endomorphism f of \mathbb{P}^2 , there exists a neighborhood V of μ_{eq} such that*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\max_{L \in (\mathbb{P}^2)^*} \text{genus}(f^{-n}(L) \cap V^c) \right) \leq \log d.$$

By [Du2, Theorem 1.1], we know that we cannot improve this result without an additional assumption. An idea is to adapt de Thelin proof in the basin of a small topological degree attracting set (see [Da,DT] for the definition). The proof of [deT4, Theorem 2] is essentially in two steps:

- (1) Controlling the number of “small” handles of $f^{-n}(L)$, which is about the same as the number of preimages staying in $U = \mathbb{P}^2 \setminus V$.
- (2) Controlling the number of “larger” handles.

Under the small topological degree assumption, we may adapt de Thelin’s proof to get that the number of “small” handles is bounded by $O(d^n)$. But the control of the number of “larger” handles does not seems to be linked with the number of preimages, so we only have that the genus of $f^{-n}(L)$ in a trapping region growth as $O(nd^n)$.

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Sandrine Daurat
University of Michigan
Ann Arbor, MI, 48109, USA
E-mail:daurat@umich.edu